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Maharam extensions of positive operators and f-modules

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Abstract. The principal result of this paper is the construction of simultaneous extensions of collections of positive linear operators between vector lattices to interval preserving operators (i.e., Maharam operators). This construction is based on some properties of so-called f-modules. The properties and structure of these extension spaces is discussed in some detail.

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1. Introduction

One of the fundamental theorems in measure theory is the classical Radon-Nikodym theorem: if μ and ν are two σ-finite σ-additive measures on a measurable space (X, Σ) satisfying $0 \leq \nu(A) \leq \mu(A)$ for all $A \in \Sigma$, then there exists a bounded Σ-measurable function m on X such that $d\nu = m d\mu$. This Radon-Nikodym theorem can also be formulated in terms of positive linear functionals on some lattice ideal E of measurable functions on some σ-finite measure space (X, Σ, λ) as follows. If φ and ψ are normal (i.e., order continuous) linear functionals on E satisfying $0 \leq \psi(u) \leq \varphi(u)$ for all $0 \leq u \in E$, then $\psi(u) = \varphi(mu)$ for some bounded measurable function m on X. Equivalently, $\psi = \varphi \circ \pi_m$ where π_m denotes the operator of multiplication by m in E. For positive linear operators such a Radon-Nikodym theorem is in general not valid. However, it will follow from the results in the present paper that, given any collection of positive operators, it is always possible to enlarge the domain space of the operators involved such that a Radon-Nikodym theorem holds for the extended operators.

The natural setting to discuss these problems is the framework of vector lattices (or, Riesz spaces). In this setting a general Radon-Nikodym type theorem has been obtained for so-called interval preserving operators (or, Maharam operators) in [7]. We briefly recall this result. Let L and M be Dedekind complete vector lattices and assume that $T : L \to M$ is a positive linear operator (so, $0 \leq u \in L$ implies that $Tu \geq 0$ in M). Such an operator is called interval preserving if $0 \leq w \leq Tu$ in M implies that $w = Tv$ for some $0 \leq v \leq u$ in L, i.e., $T[0, u] = [0, Tu]$ for all

Dedicated to the memory of our friend and colleague Pay Huijsmans.

$0 \le u \in L$). It was shown in [7] that, if $0 \le S \le T : L \to M$, with T interval preserving and normal (i.e., order continuous), then $S = T\pi$ for some $0 \le \pi \le I$ in $Z(L)$. Here $Z(L)$ denotes the center of the space L, i.e., $Z(L)$ is the algebra of abstract multiplication operators in L. Therefore, the main concern in the present paper is the construction of extensions of a given collection of positive operators from L into M to interval preserving operators from a larger space $\widehat{L}$ into M. In order to apply the above mentioned Radon-Nikodym theorem to the extended operators it will be important that the larger space $\widehat{L}$ is Dedekind complete and that the extended operators are order continuous.

To illustrate these ideas we will now discuss a typical example of such an extension space (for details see Example B in Section 6). Assume that $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ are two σ-finite measure spaces and that $L \subseteq L_0(\nu)$ and $M \subseteq L_0(\mu)$ are two ideals of measurable functions. Let $\mathcal{L}_k^+(L, M)$ denote the collection of all positive kernel operators from L into M, i.e., any $T \in \mathcal{L}_k^+(L, M)$ is given by

$$Tf(x) = \int_Y k(x, y) f(y) \mathrm{d}\nu(y),$$

for some measurable kernel $k(x, y) \ge 0$ on $X \times Y$. Now define the ideal $\widehat{L} \subseteq L_0(\mu \otimes \nu)$ by

$$\widehat{L} = \{g \in L_0(\mu \otimes \nu) : |g(x, y)| \le |f(y)| \text{ for some } f \in L\}.$$

Identifying $f \in L$ with $\mathbb{I}_X \otimes f \in \widehat{L}$, we consider L as a subspace of $\widehat{L}$. Defining $\widehat{T} : \widehat{L} \to M$ by

$$\widehat{T}g(x) = \int_Y k(x, y) g(x, y) \mathrm{d}\nu(y),$$

it follows that $\widehat{T}$ is an extension of T which is interval preserving. The purpose of the present paper is to construct such interval preserving extensions for arbitrary collections of linear positive operators between vector lattices, which we will call Maharam extensions, a terminology we will explain next.

In the paper [10], D. Maharam proved a version of the Radon-Nikodym theorem for ‘full-valued (measurable) function-valued integrals’ or, full-valued F-integrals. We recall that F-integrals are induced by integration with respect to ‘measures’ which take their values in an algebra of measurable functions. ‘Full-valued’ is the term that Maharam introduced for what we have referred to earlier as interval preserving. It is of interest to point out that in the case of numerical valued measures the integrals they determine are always trivially full-valued, but that the measures are so if they are non-atomic. It was Maharam’s insight that vector-valued versions of the results of the classical theory of integration depend in an essential way on the full-valuedness property. Whence the name Maharam operators, which are the main object of study in this paper. The result that positive operators have Maharam extensions is inspired by the corresponding extension theorem given by Maharam for F-integrals in [9] (see also [11]). In our general setting, however, the construction method of the extensions proceeds along completely different lines because of the non-availability of Maharam's slice integral method.

After fixing some notation in Section 2, we discuss some of the basic properties of Maharam operators in Section 3. Given two Archimedean Riesz spaces L and M, with M Dedekind complete, an order bounded linear operator T form L into M will be called a Maharam operator if the absolute value $|T|$ is interval preserving. The properties of such Maharam operators were first studied in [7]. First we present a simple proof of the important result that for any order continuous Maharam operator T between Dedekind complete spaces L and M there exists an algebra and lattice homomorphism h from $Z(M)$ into $Z(L)$ such that $\pi T = Th(\pi)$ for all $\pi \in Z(M)$. Using this result all of the basic properties of Maharam operators are then deduced, including the Radon-Nikodym theorem (see Proposition 3.10). The existence of such a homomorphism h shows that Maharam operators are actually $Z(M)$-linear operators with respect to an appropriate $Z(M)$-module structure on L induced by h. This point of view plays actually a key role in all the results in the present paper. Therefore, in Section 4 we study some of the relevant properties of so-called f-modules and in particular of f-modules over $Z(M)$ for some Dedekind complete space M. Given a $Z(M)$-module E we denote by $\mathcal{L}_n^{Z(M)}(E, M)$ the space of all normal $Z(M)$-linear operators from E into M. It turns out that the structure of the space $\mathcal{L}_n^{Z(M)}(E, M)$ has many properties in common with the normal dual space $L_n^\sim$ of a vector lattice L. In particular, one of the main results in Section 4 is the analogue of the well-known Nakano perfectness criterion for vector lattices in the setting of $Z(M)$-modules (see Theorem 4.9). This result plays a crucial role in the construction of the Maharam extensions.

Section 5 is devoted to the construction of the Maharam extension spaces (see Definition 5.6) for ideals of operators in $\mathcal{L}_b(L, M)$. Furthermore, the uniqueness of these extensions is discussed and several characterizations of such extension spaces are given. The main results are Theorems 5.4 and 5.9. In Section 6 we discuss a number of examples of Maharam extension spaces, in particular for kernel operators and lattice homomorphisms. Finally in Section 7 the structure of Maharam extension spaces is discussed in detail. In particular, in Theorem 7.8 a description of the Boolean algebra of band projections in such extension spaces is given. As a consequence it follows that for any positive linear operator T between Dedekind complete spaces L and M, the Boolean algebra $\mathcal{B}_T$ of all components is equal to the complete algebra generated by the components of the form PTQ, where P and Q are band projections in M and L respectively. The paper ends with a list of topics to be discussed in a subsequent paper.

Some of the main results of the present paper, in particular Theorem 5.4, where already presented by the first author at the Conference in Honor of Dorothy Maharam Stone in 1987 and have been announced in [6] without proofs. Also [6] contains a discussion of some of Maharam's work concerning her deep analysis of the theory of F-integrals and their extensions from the point of view of the present theory of positive operators.

After the completion of our paper it was pointed out to us by Anton R. Schep that similar problems for a single positive operator have been considered by G.P. Akilov, E.V. Kolesnikov and A.G. Kusraev in [1]. One of the major and important differences with the results in [1], is that in the present paper we construct Maharam extension spaces not only for a single operator but for arbitrary collections of operators.

2. Preliminaries

For the general theory of Riesz spaces (vector lattices) we refer to the books [3], [8], [12] and [13]. For any Archimedean Riesz space L we denote by $\mathcal{B}(L)$ the complete Boolean algebra of bands in L (see [8, Section 22]). By $\mathcal{P}(L)$ we denote the Boolean algebra of band projections in L (see [8, Section 30]). If L is an Archimedean Riesz space and M a Dedekind complete Riesz space, then $\mathcal{L}_b(L, M)$ denotes the Dedekind complete Riesz space of all order bounded linear operators form L into M (see e.g. [13, Section 83]). For $T \in \mathcal{L}_b(L, M)$ the null ideal of T is defined by

$$\mathcal{N}_T = \{ f \in L : |T|(|f|) = 0 \}$$

and $\mathcal{C}_T = \mathcal{N}_T^d$ is called the carrier of T. If T is order continuous, then $\mathcal{N}_T$ is a band and if in addition L is Dedekind complete, then $L = \mathcal{C}_T \oplus \mathcal{N}_T$. In this case we denote by C_T and N_T the band projections in L onto $\mathcal{C}_T$ and $\mathcal{N}_T$ respectively. As usual we denote by $\mathrm{Orth}(L)$ the f-algebra of all orthomorphisms in L (see e.g. [13, Chapter 20]), and $Z(L)$ denotes the center of L, i.e., the (order) ideal generated by the identity I in $\mathrm{Orth}(L)$. For $\pi \in \mathrm{Orth}(L)$ we consider the mapping

$$R_\pi : \mathcal{L}_b(L, M) \to \mathcal{L}_b(L, M)$$

defined by $R_\pi(T) = T\pi$ for all $T \in \mathcal{L}_b(L, M)$. Then $R_\pi \in \mathrm{Orth}\big(\mathcal{L}_b(L, M)\big)$. Indeed, if $\pi \in Z(L)$, then it is clear that $R_\pi \in Z\big(\mathcal{L}_b(L, M)\big)$. If $0 \le \pi \in \mathrm{Orth}(L)$, then $\pi \wedge nI \uparrow \pi$, π^2-uniformly (see e.g. [13, Corollary 142.8]), from which it easily follows that $T_1 \wedge T_2 = 0$ in $\mathcal{L}_b(L, M)$ implies that $T_1 \wedge (T_2\pi) = 0$. This suffices to show that $R_\pi \in \mathrm{Orth}\big(\mathcal{L}_b(L, M)\big)$ for all $\pi \in \mathrm{Orth}(L)$. Moreover, the mapping $\pi \mapsto R_\pi$ is an f-algebra homomorphism (i.e., an algebra and Riesz homomorphism) from $\mathrm{Orth}(L)$ into $\mathrm{Orth}\big(\mathcal{L}_b(L, M)\big)$. Indeed, it is clear that this mapping is a positive algebra homomorphism, hence it is a Riesz homomorphism (i.e., lattice homomorphism) as well. It follows now in particular that, for example,

$$T(\pi_1 \vee \pi_2) = (T\pi_1) \vee (T\pi_2)$$

for all $\pi_1, \pi_2 \in \mathrm{Orth}(L)$ and $0 \le T \in \mathcal{L}_b(L, M)$. Similarly, for any $\pi \in \mathrm{Orth}(M)$ we define the operator L_π from $\mathcal{L}_b(L, M)$ into itself by $L_\pi(T) = \pi T$ for all $T \in \mathcal{L}_b(L, M)$. Then $L_\pi \in \mathrm{Orth}\big(\mathcal{L}_b(L, M)\big)$ and the mapping $\pi \mapsto L_\pi$ is an f-algebra homomorphism from $\mathrm{Orth}(M)$ into $\mathrm{Orth}\big(\mathcal{L}_b(L, M)\big)$. Note that it is now clear that $|\sigma T\pi| = |\sigma| \cdot |T| \cdot |\pi|$ for all $\sigma \in \mathrm{Orth}(M)$, $\pi \in \mathrm{Orth}(L)$ and $T \in \mathcal{L}_b(L, M)$.

3. Maharam Operators

In this section we will discuss and review some of the important properties of Maharam operators which will be used throughout the paper. Actually we will also include some alternative proofs of known results, as this may be of interest in its own right. We start by recalling the following definition.

DEFINITION 3.1 *Let L and M be Archimedean Riesz spaces with M Dedekind complete. An operator $T \in \mathcal{L}_b(L, M)$ is a Maharam operator (or, has the Maharam property) if $|T|$ is interval preserving (i.e., if $0 \leq w \leq |T|u$ in M, then there exists $0 \leq v \leq u$ in L such that $w = |T|v$).*

Now we will show how a Boolean ring homomorphism $\sigma_T : \mathcal{B}(M) \to \mathcal{B}(L)$ can be associated with any Maharam operator $T \in \mathcal{L}_n(L, M)$. As above, M is assumed to be Dedekind complete. For $B \in \mathcal{B}(M)$ let

$$\sigma_T(B) = \{ f \in L : |T|(|f|) \in B \} \cap \mathcal{C}_T. \tag{3.1}$$

It it clear that $\sigma_T(B)$ is a band in L, i.e., $\sigma_T(B) \in \mathcal{B}(L)$.

LEMMA 3.2 *The mapping $\sigma_T : \mathcal{B}(M) \to \mathcal{B}(L)$ is an order continuous Boolean ring homomorphism.*

Proof. Since $\sigma_T = \sigma_{|T|}$, we may assume that $T \geq 0$. It is easy to see that $\sigma_T(\{0\}) = \{0\}$, $\sigma_T(B_1 \cap B_2) = \sigma_T(B_1) \cap \sigma_T(B_2)$ and $\sigma_T(B_1) \vee \sigma_T(B_2) \subseteq \sigma_T(B_1 \vee B_2)$ for all $B_1, B_2 \in \mathcal{B}(M)$. So it remains to show that $\sigma_T(B_1 \vee B_2) \subseteq \sigma_T(B_1) \vee \sigma_T(B_2)$ for $B_1, B_2 \in \mathcal{B}(M)$. Take $0 \leq f \in \sigma_T(B_1 \vee B_2)$, i.e., $0 \leq f \in \mathcal{C}_T$ and $Tf \in B_1 \vee B_2$. Since M is Dedekind complete, $B_1 \vee B_2 = B_1 + B_2$ and so we can write $Tf = w_1 + w_2$ with $0 \leq w_j \in B_j (j = 1, 2)$. Since T is Maharam there exists $0 \leq u_1 \leq f$ in L such that $Tu_1 = w_1$. Put $u_2 = f - u_1$. Then $u_j \in \sigma_T(B_j)$ for $j = 1, 2$ and hence $f \in \sigma_T(B_1) + \sigma_T(B_2) \subseteq \sigma_T(B_1) \vee \sigma_T(B_2)$. We may conclude that σ_T is a Boolean ring homomorphism. Finally, since $T\big(\sigma_T(B)\big) \subseteq B$ for all $B \in \mathcal{B}(M)$, it follows easily that $B_\tau \downarrow 0$ in $\mathcal{B}(M)$ implies that $\sigma_T(B_\tau) \downarrow 0$ in $\mathcal{B}(L)$, i.e., σ_T is order continuous. □

Now assume that L is Dedekind complete as well. Then the Boolean algebra $\mathcal{B}(L)$ of bands in L is isomorphic with the Boolean algebra $\mathcal{P}(L)$ of band projections in L, and similarly for $\mathcal{B}(M)$ and $\mathcal{P}(M)$. Therefore, the Boolean ring homomorphism σ_T constructed above induces a corresponding Boolean ring homomorphism $h_T : \mathcal{P}(M) \to \mathcal{P}(L)$. Note that $h_T(I) = C_T$, so $Th_T(I) = T$. Moreover, since $T\big(\sigma_T(B)\big) \subseteq B$ for all $B \in \mathcal{B}(M)$, it follows that $PTh_T(P) = Th_T(P)$ for all $P \in \mathcal{P}(M)$. Applying this identity with P replaced by $I - P$ and using that $Th_T(I) = T$, we find that

$$PT = Th_T(P) \quad \text{and} \quad h_T(P) \leq C_T, \quad \forall P \in \mathcal{P}(M). \tag{3.2}$$

Via a standard argument, involving Freudenthal's spectral theorem, it now follows that h_T extends to an order continuous f-algebra homomorphism from $Z(M)$ into $Z(L)$, which we denote by h_T again. It is clear from (3.2) that

$$\pi T = Th_T(\pi) \quad \text{and} \quad h_T(\pi)C_T = h_T(\pi), \quad \forall \pi \in Z(M). \tag{3.3}$$

We thus have recovered via an alternative approach the result of [7], Theorem 2.3. For sake of reference we formulate the result in the following proposition.

PROPOSITION 3.3 *([7]) Let L and M be Dedekind complete Riesz spaces and $T \in \mathcal{L}_n(L, M)$.*

(i) If T is a Maharam operator, then there exists a unique f-algebra homomorphism $h_T\colon Z(M) \to Z(L)$ satisfying (3.3)

(ii) If there exists an f-algebra homomorphism $h\colon Z(M) \to Z(L)$ such that $\pi T = Th(\pi)$ for all $\pi \in Z(M)$, then T is a Maharam operator.

Proof. (i) Only the uniqueness statement needs some explanation. To this end observe that if $\pi_1, \pi_2 \in Z(L)$ such that $T\pi_1 = T\pi_2$ and $\pi_j C_T = \pi_j$ $(j = 1, 2)$, then $\pi_1 = \pi_2$. Indeed, $T(\pi_1 - \pi_2) = 0$ implies that $|T| \cdot |\pi_1 - \pi_2| = 0$ and hence

$$|\pi_1 - \pi_2| = |\pi_1 C_T - \pi_2 C_T| = |\pi_1 - \pi_2| C_T = 0.$$

(ii) First note that for $\pi \in Z(M)$ the mappings $S \mapsto Sh(\pi)$ and $S \mapsto \pi S$ are center operators in $\mathcal{L}_n(L, M)$. Hence the set

$$\{\, S \in \mathcal{L}_n(L, M) \colon \pi S = Sh(\pi) \,\}$$

is a band in $\mathcal{L}_n(L, M)$. Consequently, the hypothesis on T implies that $\pi|T| = |T|h(\pi)$ for all $\pi \in Z(M)$. Now assume that $0 \le u \in L$ and $0 \le w \le |T|u$ in M. Since M is Dedekind complete, there exists $0 \le \pi \le I$ in $Z(M)$ such that $w = \pi|T|u = |T|\big(h(\pi)u\big)$. Moreover, $0 \le h(\pi) \le h(I) \le I$ and so $0 \le h(\pi)u \le u$. This shows that $|T|$, and hence T, is a Maharam operator. □

Motivated by the above results we introduce some notation. Let L and M be Archimedean Riesz spaces with M Dedekind complete and let $h\colon Z(M) \to Z(L)$ be an f-algebra homomorphism. Then we define

$$\mathcal{L}_n^h(L, M) = \{\, T \in \mathcal{L}_n(L, M) \colon \pi T = Th(\pi) \quad \forall \pi \in Z(M) \,\}.$$

LEMMA 3.4 *$\mathcal{L}_n^h(L, M)$ is a band in $\mathcal{L}_n(L, M)$.*

Proof. For $\pi \in Z(M)$ and $\sigma \in Z(L)$ we define the operators $L_\pi, R_\sigma \in Z(\mathcal{L}_n(L, M))$ by $L_\pi(T) = \pi T$ and $R_\sigma(T) = T\sigma$ respectively for all $T \in \mathcal{L}_n(L, M)$. Then

$$\mathcal{L}_n^h(L, M) = \bigcap \big\{ \ker(L_\pi - R_{h(\pi)}) \colon \pi \in Z(M) \big\}.$$

Since the kernel of any center operator is a band it now follows immediately that $\mathcal{L}_n^h(L, M)$ is a band in $\mathcal{L}_n(L, M)$. □

COROLLARY 3.5 *If L and M are Dedekind complete Riesz spaces and $T \in \mathcal{L}_n(L, M)$ is a Maharam operator, then all operators $S \in \{T\}^{dd}$ are Maharam.*

Proof. Let $h = h_T$ be the f-algebra homomorphism associated with T as in Proposition 3.3(i). Then $T \in \mathcal{L}_n^h(L, M)$ and so $\{T\}^{dd} \subseteq \mathcal{L}_n^h(L, M)$, as $\mathcal{L}_n^h(L, M)$ is a band by Lemma 3.4. It follows from Proposition 3.3 (ii) that all operators in $\mathcal{L}_n^h(L, M)$ are Maharam. □

Next we will discuss the lattice structure of $\mathcal{L}_n^h(L, M)$ in some more detail. Note that if $M = \mathbb{R}$ (and so $Z(M) = \mathbb{R}$ as well), then $L_n^\sim = \mathcal{L}_n(L, \mathbb{R}) = \mathcal{L}_n^h(L, \mathbb{R})$, where $h(\alpha) = \alpha I$ for all $\alpha \in \mathbb{R}$. The space $L_n^\sim$ has a number of properties which are not shared by $\mathcal{L}_n(L, M)$ in general. For example, $\psi \perp \varphi$ in $L_n^\sim$ is equivalent to $\mathcal{C}_\psi \perp \mathcal{C}_\varphi$ and there is a natural correspondence between bands in $L_n^\sim$ and in L. We will see that in this respect there are a number of analogies between $\mathcal{L}_n^h(L, M)$ and $L_n^\sim$. We start with a lemma.

LEMMA 3.6 *Let L and M be Dedekind complete Riesz spaces and $h\colon Z(M) \to Z(L)$ an f-algebra homomorphism. Suppose that $S, T \in \mathcal{L}_n^h(L, M)$ with $S \wedge T = 0$. If $0 \le u \in L$ and $0 \le w \in M$ such that $Tu \in \{w\}^{dd}$, then there exists $0 \le v \le u$ in L such that $Tv + S(u - v) \le w$.*

Proof. Define

$$\mathcal{Q} = \Big\{ Q \in \mathcal{P}(M)\colon \ \exists\, 0 \le v \le u \text{ in } L \text{ such that } Q\big[\, Tv + S(u - v) \,\big] \le w \Big\}.$$

It is clearly sufficient to show that $I \in \mathcal{Q}$. The proof of this uses the following two observations.

(i) If $0 \ne P \in \mathcal{P}(M)$ then there exists $0 \ne Q \in \mathcal{Q}$ such that $Q \le P$. Indeed, if $Pw = 0$ then $PTu = 0$ and so $P \in \mathcal{Q}$ (take $v = u$). Now assume that $Pw > 0$. Then $S \wedge T = 0$ implies that there exists $0 \le v \le u$ such that

$$z = \Big\{ Pw - \big[\, Tv + S(u - v) \,\big] \Big\}^+ > 0.$$

Let Q be the band projection in M onto $\{z\}^{dd}$. Then $0 \ne Q \le P$ and

$$Q\big[\, Tv + S(u - v) \,\big] \le Pw \le w,$$

so $Q \in \mathcal{Q}$. This proves the claim.

(ii) If $\{Q_\alpha : \alpha \in A\}$ is a disjoint system in $\mathcal{Q}$, then $Q = \bigvee_\alpha Q_\alpha \in \mathcal{Q}$. Indeed, let $0 \le v_\alpha \le u$ be such that $Q_\alpha\big[Tv_\alpha + S(u - v_\alpha)\big] \le w$. Define

$$v = \bigvee_\alpha h(Q_\alpha) v_\alpha.$$

Since h is an f-algebra homomorphism it follows that $\big\{h(Q_\alpha)\colon \alpha \in A\}$ is a disjoint system in $\mathcal{P}(L)$. Hence $h(Q_\alpha)v = h(Q_\alpha)v_\alpha$. Now

$$\begin{aligned} Q_\alpha\big[Tv + S(u-v)\big] &= Th(Q_\alpha)v_\alpha + S\big(h(Q_\alpha)u - h(Q_\alpha)v_\alpha\big) = \\ &= Q_\alpha\big[Tv_\alpha + S(u-v_\alpha)\big] \le w \end{aligned}$$

for all $\alpha \in A$. This implies that $Q\big[Tv + S(u-v)\big] \le w$, hence $Q \in \mathcal{Q}$ and the claim is proved.

Now take a maximal disjoint system $\{Q_\alpha\}$ in $\mathcal{Q}$. From (i) it follows that $\{Q_\alpha\}$ is a maximal disjoint system in $\mathcal{P}(M)$, so $\bigvee_\alpha Q_\alpha = I$, and now (ii) implies that $I \in \mathcal{Q}$. $\square$

PROPOSITION 3.7 *Let L and M be Archimedean Riesz spaces with M Dedekind complete and let $h : Z(M) \to Z(L)$ be an f-algebra homomorphism. For $S, T \in \mathcal{L}_n^h(L, M)$ the following two statements are equivalent:*

(i) $S \perp T$; (ii) $\mathcal{C}_S \perp \mathcal{C}_T$.

Proof. The implication (ii) $\Rightarrow$ (i) holds for any two order continuous operators. Indeed, $\mathcal{C}_S \perp \mathcal{C}_T$ implies that $\mathcal{C}_S \subseteq \mathcal{N}_T$ and so the order continuous operator $|S| \wedge |T|$ vanishes on the order dense ideal $\mathcal{C}_S \oplus \mathcal{N}_S$. Hence $|S| \wedge |T| = 0$.

For the proof of (ii) $\Rightarrow$ (i) we may assume that $S, T \ge 0$. Moreover, we first assume in addition that L is Dedekind complete as well. Take $0 \le u \in \mathcal{C}_T$ and let $w = Tu$. From the above lemma it follows that for $n = 1, 2, \dots$ there exists $0 \le v_n \le u$ such that

$$Tv_n + S(u - v_n) \le 2^{-n}w.$$

Define $z_n = \sup\{v_k : k \ge n\}$. Since T is order continuous it follows that

$$0 \le Tz_n \le \sum_{k=n}^{\infty} 2^{-k}w = 2^{-n+1}w.$$

Let $z = \inf_n z_n$. Then $z_n \downarrow z$, so $Tz_n \downarrow Tz$ and hence $Tz = 0$. Now $0 \le z \le u$ implies $0 \le z \in \mathcal{C}_T$, so $z = 0$, i.e., $z_n \downarrow 0$. Therefore $0 \le u - z_n \uparrow u$, and hence $S(u - z_n) \uparrow Su$. Since

$$0 \le S(u - z_n) \le S(u - v_n) \le 2^{-n}w$$

for all $n = 1, 2, \dots$, it follows that $Su = 0$, i.e. $u \in \mathcal{N}_S$. We have shown that $\mathcal{C}_T \subseteq \mathcal{N}_S$, i.e., $\mathcal{C}_T \perp \mathcal{C}_S$. Now we return to the general situation where L is only assumed to be Archimedean. Let $L^\wedge$ be the Dedekind completion of L. Every operator $R \in \mathcal{L}_n(L, M)$ has a unique extension $\widehat{R} \in \mathcal{L}_n(L^\wedge, M)$ and the mapping $R \mapsto \widehat{R}$ is a Riesz isomorphism from $\mathcal{L}_n(L, M)$ onto $\mathcal{L}_n(L^\wedge, M)$. Similarly, every $\pi \in Z(L)$ has a unique extension $\widehat{\pi} \in Z(L^\wedge)$ and the mapping $\pi \mapsto \widehat{\pi}$ defines an

f-algebra isomorphic embedding of $Z(L)$ into $Z(L^\wedge)$. Therefore, the f-algebra homomorphism $h\colon Z(M) \to Z(L)$ can be considered as an f-algebra homomorphism from $Z(M)$ into $Z(L^\wedge)$. Now it is clear that the above isomorphism $R \mapsto \widehat{R}$ maps $\mathcal{L}_n^h(L, M)$ onto $\mathcal{L}_n^h(L^\wedge, M)$. Furthermore it is easy to check that for all $R \in \mathcal{L}_n(L, M)$ we have $\mathcal{N}_R = \mathcal{N}_{\widehat{R}} \cap L$ and $\mathcal{C}_R = \mathcal{C}_{\widehat{R}} \cap L$.

Now assume that $S, T \in \mathcal{L}_n^h(L, M)$ such that $S \perp T$. Then $\widehat{S} \perp \widehat{T}$ in $\mathcal{L}_n^h(L^\wedge, M)$ and by the first part of the proof it follows that $\mathcal{C}_{\widehat{S}} \perp \mathcal{C}_{\widehat{T}}$. Consequently $\mathcal{C}_S \perp \mathcal{C}_T$, by which the proof of the proposition is complete. □

As mentioned before, in case $M = \mathbb{R}$ the result of the above proposition for the space $L_n^\sim$ is well-known (see e.g. [13], Theorem 90.6 or [12], Theorem 1.4.11) and goes back to H. Nakano. Our proof of the above proposition is patterned after the case that $M = \mathbb{R}$.

Now we will discuss some consequences of Proposition 3.7. For convenience we denote $E = \mathcal{L}_n^h(L, M)$, where L, M and h are as in the above proposition. For any band $B \subseteq E$ we define the absolute null ideal of B by

$$\mathcal{N}_B = \big\{\, f \in L\colon |T|(|f|) = 0 \quad \forall T \in B \,\big\}.$$

It is clear that $\mathcal{N}_B$ is a band in L and that $\mathcal{N}_B = \bigwedge\{\mathcal{N}_T\colon T \in B\}$ in $\mathcal{B}(L)$. The carrier of B is now defined by

$$\mathcal{C}_B = \mathcal{N}_B^d.$$

Note that $\mathcal{C}_B = \bigvee\{\mathcal{C}_T\colon T \in B\}$ in $\mathcal{B}(L)$. It is easy to see that if $B = \{T\}^{dd}$ for some $T \in E$, then $\mathcal{N}_B = \mathcal{N}_T$ and $\mathcal{C}_B = \mathcal{C}_T$.

PROPOSITION 3.8 *With the notation introduced above, the following holds.*
(i) The mapping $\tau\colon \mathcal{B}(E) \to \mathcal{B}(L)$ defined by $\tau(B) = \mathcal{C}_B$, is an injective Boolean ring homomorphism with $\tau\big(\mathcal{B}(E)\big) = \mathcal{B}(\mathcal{C}_E)$.
(ii) For all $B \in \mathcal{B}(E)$ we have

$$B = \big\{\, T \in \mathcal{L}_n^h(L, M)\colon \mathcal{C}_T \subseteq \mathcal{C}_B \,\big\}.$$

In particular, if $S, T \in \mathcal{L}_n^h(L, M)$ then $S \in \{T\}^{dd}$ if and only if $\mathcal{C}_S \subseteq \mathcal{C}_T$.

Proof. (i) If $B_1, B_2 \in \mathcal{B}(E)$ then it is easy to see that $\mathcal{N}_{B_1 \vee B_2} = \mathcal{N}_{B_1} \cap \mathcal{N}_{B_2}$, hence $\mathcal{C}_{B_1 \vee B_2} = \mathcal{C}_{B_1} \vee \mathcal{C}_{B_2}$, i.e., $\tau(B_1 \vee B_2) = \tau(B_1) \vee \tau(B_2)$. Now assume that $B_1, B_2 \in \mathcal{B}(E)$ are such that $B_1 \cap B_2 = \{0\}$. Then $S \perp T$ for all $S \in B_1$ and $T \in B_2$, so it follows from Proposition 3.7 that $\mathcal{C}_S \perp \mathcal{C}_T$ for all $S \in B_1$, $T \in B_2$. Hence,

$$\mathcal{C}_{B_1} = \bigvee\{\, \mathcal{C}_S\colon S \in B_1 \,\} \perp \bigvee\{\, \mathcal{C}_T\colon T \in B_2 \,\} = \mathcal{C}_{B_2},$$

i.e., $\tau(B_1) \wedge \tau(B_2) = 0$. From these two properties of τ it follows that τ is a Boolean ring homomorphism. If $B \in \mathcal{B}(E)$ is such that $\tau(B) = 0$, then $\mathcal{N}_B = L$ and so $B = \{0\}$. This shows that τ is injective.

Now take $A \in \mathscr{B}(\mathcal{C}_E)$ and define

$$B = \left\{ T \in \mathcal{L}_n^h(L, M) \colon \mathcal{C}_T \subseteq A \right\}.$$

Then $B \in \mathscr{B}(E)$ and $\tau(B) \subseteq A$. We claim that $\tau(B) = A$. Indeed, let $A_1 = A \cap \tau(B)^d$ and define

$$B_1 = \left\{ T \in \mathcal{L}_n^h(L, M) \colon \mathcal{C}_T \subseteq A_1 \right\}.$$

Then $B_1 \in \mathscr{B}(E)$ and $\tau(B_1) \subseteq A_1$. Consequently, $\tau(B_1 \cap B) = \tau(B_1) \cap \tau(B) = \{0\}$, so $B_1 \cap B = \{0\}$. It follows from $A_1 \subseteq A$ that $B_1 \subseteq B$, and hence $B_1 = \{0\}$. Now assume that $A_1 \neq \{0\}$. Since $A_1 \subseteq \mathcal{C}_E$ there exists $0 \leq T \in \mathcal{L}_n^h(L, M)$ such that $T_{|A_1} \neq 0$. Let T_1 be the minimal positive extension of the restriction of T to A_1 (cf. [13], Theorem 83.7). Then $T_1 \neq 0$ and $A_1^d \subseteq \mathcal{N}_{T_1}$ i.e., $\mathcal{C}_{T_1} \subseteq A_1$. Hence $0 \neq T_1 \in B_1$, which is a contradiction. We may conclude therefore that $A_1 = \{0\}$, i.e., that $A \subseteq \tau(B)$ and so $\tau(B) = A$. This shows that $\tau\big(\mathscr{B}(E)\big) = \mathscr{B}(\mathcal{C}_E)$.
(ii) Take $B \in \mathscr{B}(E)$ and let

$$B_0 = \left\{ T \in \mathcal{L}_n^h(L, M) \colon \mathcal{C}_T \subseteq \mathcal{C}_B \right\}.$$

From the second part of the proof of (i) above, applied to $A = \mathcal{C}_B$, it follows that $B_0 \in \mathscr{B}(E)$ satisfies $\tau(B_0) = \mathcal{C}_B$. Since τ is injective, this implies that $B = B_0$. □

REMARK 3.9 *Consider the same situation as above but assume in addition that L is Dedekind complete as well. For $B \in \mathscr{B}(E)$ we denote by N_B and C_B the band projections in L onto $\mathcal{N}_B$ and $\mathcal{C}_B$ respectively. Note that $T = TC_B$ for all $T \in B$ and that $C_T \leq C_E \leq h(I)$ for all $T \in E = \mathcal{L}_n^h(L, M)$.*

Given $B \in \mathscr{B}(E)$ define $h_B \colon Z(M) \to Z(L)$ by $h_B(\pi) = h(\pi)C_B$ for all $\pi \in Z(M)$. It is clear that h_B is an f-algebra homomorphism and that $B \subseteq \mathcal{L}_n^{h_B}(L, M)$. Actually we have $B = \mathcal{L}_n^{h_B}(L, M)$. Indeed, if $T \in \mathcal{L}_n^{h_B}(L, M)$ then $C_T \leq h_B(I) = C_B$, and so $T \in B$ by (ii) of the above proposition.

The following proposition is another consequence of Proposition 3.7.

PROPOSITION 3.10 *Let L and M be Dedekind complete Riesz spaces and $h \colon Z(M) \to Z(L)$ an f-algebra homomorphism. Suppose that $0 \leq T \in \mathcal{L}_n^h(L, M)$.*
(i) If S is a component of T $\big($i.e., $S \wedge (T - S) = 0\big)$, then $S = TC_S$.
(ii) If $0 \leq S \leq T$ then there exists $0 \leq \pi \leq I$ in $Z(L)$ such that $S = T\pi$.

Proof. (i) From Proposition 3.7 we know that $C_S C_{T-S} = 0$ and it is clear that $S = SC_S$ and $T - S = (T - S)C_{T-S}$. Hence,

$$TC_S = SC_S + (T - S)C_S = S + (T - S)C_{T-S}C_S = S.$$

(ii) This follows from (i) in combination with the Freudenthal spectral theorem. □

Note that it follows in particular from this proposition that for every $T \in \mathcal{L}_n^h(L, M)$ we have

$$T^+ = |T|C_{T^+} = TC_{T^+}, T^- = |T|C_{T^-} = -TC_{T^-}.$$

A combination of the results above with Proposition 3.3 yields the Hahn decomposition theorem and the Radon-Nikodym theorem for Maharam operators as obtained in [7] (Theorems 2.5 and 3.1). We note that, conversely, the above results can also be derived from [7]. However, the alternative appraoch to these results via Proposition 3.7, besides being of independent interest, shows clearly the analogy with the properties of the space $L_n^\sim$.

We end this section with a result on Maharam operators which in some sense justifies our approach to the construction of 'Maharam extensions' in the later sections of the paper.

PROPOSITION 3.11 *Let L and M be Dedekind complete Riesz spaces and suppose that $\mathcal{J} \subseteq \mathcal{L}_n(L, M)$ is an ideal which consists of Maharam operators. Then there exists an f-algebra homomorphism $h\colon Z(M) \to Z(L)$ such that $\mathcal{J} \subseteq \mathcal{L}_n^h(L, M)$.*

Proof. We start with three observations. If $S, T \in \mathcal{J}$, then: (1) $S \perp T$ implies that $C_S C_T = 0$; (2) $C_S \leq C_T$ implies that $S \in \{T\}^{dd}$; (3) SC_T is the component of S in $\{T\}^{dd}$. Indeed, the first two statements follow immediately from an application of Proposition 3.3 to $|S| + |T|$ in combination with Proposition 3.7 and Proposition 3.8 (ii). The third statement is an easy consequence of (1) and (2).

Let $\{T_\alpha\}$ be a maximal disjoint system in $\mathcal{J}$. We denote the carrier projection of T_α by C_α. From (1) above it follows that $C_\alpha C_\beta = 0$ whenever $\alpha \neq \beta$. For each α there exists, by Proposition 3.3 and Lemma 3.4, an f-algebra homomorphism $h_\alpha\colon Z(M) \to Z(L)$ satisfying $\pi T = Th_\alpha(\pi)$ for all $T \in \{T_\alpha\}^{dd}$ and all $\pi \in Z(M)$. Without loss of generality we may assume that $h_\alpha(I) = C_\alpha$ (cf. Remark 3.9). Then $h_\alpha(\pi) \perp h_\beta(\pi)$ for all $\pi \in Z(M)$ whenever $\alpha \neq \beta$. Now take $0 \leq \pi \in Z(M)$. Since $0 \leq \pi \leq \lambda I$ for some $0 \leq \lambda \in \mathbb{R}$, we have $0 \leq h_\alpha(\pi) \leq \lambda C_\alpha$ for all α, hence

$$h(\pi) = \bigvee_\alpha h_\alpha(\pi)$$

is well defined in $Z(L)$. This defines a mapping $h\colon Z(M)^+ \to Z(L)^+$ and it is easy to check that h extends to an f-algebra homomorphism $h\colon Z(M) \to Z(L)$. We claim that $\pi T = Th(\pi)$ for all $\pi \in Z(M)$ and all $T \in \mathcal{J}$. Indeed, it is sufficient to show this for $0 \leq \pi \in Z(M)$ and $0 \leq T \in \mathcal{J}$. Since $\{T_\alpha\}$ is a maximal disjoint system in $\mathcal{J}$, it follows from (3) above that $TC_\alpha \in \{T_\alpha\}^{dd}$ and $T = \bigvee_\alpha TC_\alpha$. Using that left and right multiplication in $\mathcal{L}_n(L, M)$ by center operators in $Z(M)$

and $Z(L)$ respectively are itself center operators in $\mathcal{L}_n(L, M)$, we find that

$$\begin{aligned}\pi T &= \pi \bigvee_\alpha TC_\alpha = \bigvee_\alpha \pi(TC_\alpha) = \bigvee_\alpha (TC_\alpha)h_\alpha(\pi) = \\ &= \bigvee_\alpha (TC_\alpha)h(\pi) = \Big[\bigvee_\alpha TC_\alpha\Big]h(\pi) = Th(\pi).\end{aligned}$$

This proves the claim and the proof of the proposition is complete. □

COROLLARY 3.12 *Same situation as in Proposition 3.11. Then the band generated by $\mathcal{J}$ in $\mathcal{L}_n(L, M)$ consists of Maharam operators.*

REMARK 3.13 *Let $\mathcal{J}$ be an ideal in $\mathcal{L}_n(L, M)$ consisting of Maharam operators as in Proposition 3.11, and let $h\colon Z(M) \to Z(L)$ be the f-algebra homomorphism as constructed in the proof of the proposition. Let $B = \mathcal{J}^{dd}$, the band generated by $\mathcal{J}$. Then $\mathcal{J} \subseteq B \subseteq \mathcal{L}_n^h(L, M)$. Since $\{T_\alpha\}$ is a maximal disjoint system in B as well, it follows that $C_B = \bigvee_\alpha C_\alpha$. Hence $h(I) = \bigvee_\alpha C_\alpha = C_B$. This implies that actually $B = \mathcal{L}_n^h(L, M)$ (cf. Remark 3.9).*

4. f-Modules

The results in the previous section indicate that it will be convenient to consider Maharam operators in the framework of so-called f-modules. Although in the sequel we will deal only with special f-modules ($Z(M)$-modules), we start with the general definition and some elementary properties.

Let A be an Archimedean f-algebra and E an Archimedean Riesz space.

DEFINITION 4.1 *E is called a left f-module over A if:*

(i) E is a left-module over A with respect to a left multiplication $(a, f) \mapsto a \cdot f$ from $A \times E$ into E, i.e., $(a+b)\cdot f = a\cdot f + b\cdot f$, $a\cdot(f+g) = a\cdot f + a\cdot g$, $a\cdot(b\cdot f) = (ab)\cdot f$ and $a(\lambda f) = (\lambda a)\cdot f = \lambda(a\cdot f)$ for all $a, b \in A$, $f, g \in E$ and $\lambda \in \mathbb{R}$;

(ii) $a \cdot f \geq 0$ whenever $0 \leq a \in A$ and $0 \leq f \in E$;

(iii) $f \perp g$ in E implies that $a \cdot f \perp g$ in E for all $a \in A$.

The definition of a right f-module is similar. Here we will deal mainly with left f-modules and call these simply f-modules (unless explicitly stated otherwise).

REMARK 4.2 *Suppose that E is an f-module over the f-algebra A. For $a \in A$ define $\pi_a(f) = a \cdot f$ for all $f \in E$. From the above definition it is clear that $\pi_a \in$* Orth(E). *The mapping $h\colon A \to$* Orth(E) *is a positive algebra homomorphism and hence h is a Riesz homomorphism (indeed, $a \perp b$ in A implies that $ab = 0$, so $\pi_a\pi_b = 0$ and hence $\pi_a \perp \pi_b$ in* Orth(E)*).*

Conversely, if E is an Archimedean Riesz space, A an Archimedean f-algebra and $h\colon A \to \mathrm{Orth}(E)$ is an f-algebra homomorphism, then h induces an f-module structure over A on E by setting $a \cdot f = h(a)f$ for all $f \in E$ and all $a \in A$.

In the following lemma we list some elementary properties of f-modules, which follow easily from the first part of the above remark. We leave the details to the reader.

LEMMA 4.3 *Let E be an f-module over A.*
(i) $(a \vee b)f = (af) \vee (bf)$ and $(a \wedge b)f = (af) \wedge (bf)$ for all $a, b \in A$, $0 \leq f \in E$.
(ii) $|af| = |a| \cdot |f|$ for all $a \in A$, $f \in E$.
(iii) If $a \perp b$ in A, then $af \perp bg$ for all $f, g \in E$.

We stress that in Definition 4.1, even if the f-algebra A has a unit element e, we do *not* assume that $e \cdot f = f$ for all $f \in E$. However, it is easy to see that if $e \in A$ is a unit element, then the mapping $f \mapsto e \cdot f$ in E is a band projection. If $e \cdot f = f$ for all $f \in E$, then E is called a unital f-module.

Furthermore we note that if $\mathcal{J} \subseteq E$ is a band, or more generally a uniformly closed ideal in the f-module E over A, then $f \in \mathcal{J}$ implies that $af \in \mathcal{J}$ for all $a \in A$, and so $\mathcal{J}$ inherits the f-module structure, i.e., $\mathcal{J}$ is an f-submodule of E. If A has a unit element $0 < e \in A$ which is also a strong order unit in A, then every ideal $\mathcal{J} \subseteq E$ is an f-submodule.

Let A be an Archimedean f-algebra and let E and F be f-modules over A. A linear mapping $T\colon E \to F$ is called *A-linear* if $T(a \cdot f) = a \cdot Tf$ for all $f \in E$ and all $a \in A$. We define

$$\mathcal{L}_b^A(E, F) = \{ T \in \mathcal{L}_b(E, F)\colon T \text{ is } A\text{-linear} \},$$

and $\mathcal{L}_n^A(E, F) = \mathcal{L}_b^A(E, F) \cap \mathcal{L}_n(E, F)$.

LEMMA 4.4 *If E and F are f-modules over A, with F Dedekind complete, then $\mathcal{L}_b^A(E, F)$ is a band in $\mathcal{L}_b(E, F)$, and $\mathcal{L}_n^A(E, F)$ is a band in $\mathcal{L}_n(E, F)$.*

Proof. It suffices to prove the first statement. For $a \in A$ we define $\pi_a \in \mathrm{Orth}(E)$ by $\pi_a f = af$ for all $f \in E$ and we define $\sigma_a \in \mathrm{Orth}(F)$ by $\sigma_a g = ag$ for all $g \in F$. Now define the operators R_a and L_a from $\mathcal{L}_b(E, F)$ into itself by

$$R_a(T) = T\pi_a,\ L_a(T) = \sigma_a T \quad \forall T \in \mathcal{L}_b(E, F).$$

Then $R_a, L_a \in \mathrm{Orth}\big(\mathcal{L}_b(L, M)\big)$ (see Section 2) and it is clear that

$$\begin{aligned} \mathcal{L}_b^A(E, F) &= \{ T \in \mathcal{L}_b(E, F)\colon (R_a - L_a)(T) = 0 \quad \forall a \in A \} \\ &= \bigcap \{ \ker(R_a - L_a)\colon a \in A \}. \end{aligned}$$

Since the kernel $\ker(R_a - L_a)$ of the orthomorphism $R_a - L_a$ is a band, it follows that $\mathcal{L}_b^A(E, F)$ is a band as well. □

Next we will give some typical examples of f-modules and discuss the relation with the previous section.

EXAMPLES 4.5 *(1) Let L be an Archimedean Riesz space. Then L has a natural f-module structure over* $\mathrm{Orth}(L)$ *by setting $\pi \cdot f = \pi(f)$ for all $f \in L$ and $\pi \in$* $\mathrm{Orth}(L)$. *Similarly L is an f-module over $Z(L)$. If not stated otherwise we will tacidly consider a Riesz space L as an f-module over $Z(L)$ in this natural way.*
(2) Let L be an Archimedean Riesz space and suppose that F is a Dedekind complete f-module over the f-algebra A. Then $\mathcal{L}_b(L, F)$ has a natural f-module structure, defining $a \cdot T$ for $a \in A$ and $T \in \mathcal{L}_b(L, F)$ by

$$(a \cdot T)f = a \cdot Tf \quad \forall f \in L.$$

Since $\mathcal{L}_n(L, F)$ is a band in $\mathcal{L}_b(L, F)$, it is clear that $\mathcal{L}_n(L, F)$ is an f-submodule of $\mathcal{L}_b(L, F)$. In particular, if M is a Dedekind complete Riesz space, then $\mathcal{L}_b(L, M)$ and $\mathcal{L}_n(L, M)$ have natural f-module structures over $Z(M)$. Similarly, if E and F are f-modules over A, with F Dedekind complete, then $\mathcal{L}_b^A(E, F)$ and $\mathcal{L}_n^A(E, F)$ are in this way f-modules over A.
(3) Let L and M be Archimedean Riesz spaces with M Dedekind complete. Suppose that $h\colon Z(M) \to Z(L)$ is an f-algebra homomorphism. As observed in Remark 4.2, h induces on L an f-module structure over $Z(M)$ with $\pi \cdot f = h(\pi)f$ for all $f \in L$ and $\pi \in Z(M)$. Using the notation of the previous section, it is clear that $\mathcal{L}_n^{Z(M)}(L, M) = \mathcal{L}_n^h(L, M)$.

Conversely, suppose that E is an f-module over $Z(M)$ and let $h\colon Z(M) \to$ $\mathrm{Orth}(E)$ *be the corresponding f-algebra homomorphism as in Remark 4.2. Since $h(I)$ is a band projection in E, it follows that h maps $Z(M)$ into $Z(E)$, and $\mathcal{L}_n^{Z(M)}(E, M) = \mathcal{L}_n^h(E, M)$. Consequently the results of Propositions 3.7, 3.8 and 3.10 can be applied to $\mathcal{L}_n^{Z(M)}(E, M)$. Moreover, all operators in $\mathcal{L}_n^{Z(M)}(E, M)$ are Maharam operators.*
(4) Let L and M be Dedekind complete Riesz spaces and suppose that $T \in \mathcal{L}_n(L, M)$ is a Maharam operator. Then it follows from Proposition 3.3 that there exists an f-module structure over $Z(M)$ on L such that $T \in \mathcal{L}_n^{Z(M)}(L, M)$. Actually, it follows from Proposition 3.11 (and Remark 3.13) that, if $B \subseteq \mathcal{L}_n(L, M)$ is a band consisting of Maharam operators, then there exists an f-module structure on L such that $B = \mathcal{L}_n^{Z(M)}(L, M)$.

Our next objective is to discuss a construction for f-modules over $Z(M)$ which in the case $M = \mathbb{R}$ corresponds to the embedding into the order bidual (see e.g. [13], Section 109). In particular we are interested in the analogue of perfect Riesz spaces (see [13], Section 110) in the context of $Z(M)$-modules. These results will play an essential role in the construction of Maharam extentions in the next section of this paper.

Let M be a Dedekind complete Riesz space and suppose that E is an f-module over $Z(M)$. By Lemma 4.4, $\mathcal{L}_n^{Z(M)}(E, M)$ is a band in $\mathcal{L}_n(E, M)$ and, as observed above, $\mathcal{L}_n^{Z(M)}(E, M)$ is an f-module over $Z(M)$, where for $\pi \in Z(M)$ and $T \in \mathcal{L}_n^{Z(M)}(E, M)$ the product $\pi \cdot T$ is simply defined by $(\pi \cdot T)f = \pi(Tf)$ for all $f \in E$. Let $\mathcal{J}$ be an ideal in $\mathcal{L}_n^{Z(M)}(E, M)$; as observed before, then $\mathcal{J}$ is an f-submodule of $\mathcal{L}_n^{Z(M)}(E, M)$. For $f \in E$ we define the mapping $\tilde{f}\colon \mathcal{J} \to M$ by $\tilde{f}(T) = Tf$ for all $T \in \mathcal{J}$.

LEMMA 4.6 *$\tilde{f} \in \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ for all $f \in E$.*

Proof. Since $\tilde{f} = (f^+)^\sim - (f^-)^\sim$ and $(f^+)^\sim, (f^-)^\sim \geq 0$, it is clear that $\tilde{f} \in \mathcal{L}_b(\mathcal{J}, M)$. Now suppose that $T_\tau \downarrow 0$ in $\mathcal{J}$. Then $T_\tau \downarrow 0$ in $\mathcal{L}_n(E, M)$ and so $T_\tau|f| \downarrow 0$ in M. Since $|\tilde{f}(T_\tau)| \leq T_\tau|f|$, this implies that $\inf_\tau |\tilde{f}(T_\tau| = 0$. Hence $\tilde{f} \in \mathcal{L}_n(\mathcal{J}, M)$. Moreover, if $\pi \in Z(M)$ and $T \in \mathcal{J}$, then $\tilde{f}(\pi T) = (\pi T)f = \pi \tilde{f}(T)$, so $\tilde{f}$ is $Z(M)$-linear. □

Now define the mapping $\alpha\colon E \to \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ by $\alpha(f) = \tilde{f}$ for all $f \in E$.

LEMMA 4.7 *α is an order continuous $Z(M)$-linear Riesz homomorphism from E into $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$.*

Proof. It is clear that α is a positive linear mapping. Now suppose that $f_\tau \downarrow 0$ in E and $0 \leq T \in \mathcal{J}$. Since T is order continuous it follows that $Tf_\tau \downarrow 0$ in M, i.e., $\tilde{f}_\tau(T) \downarrow 0$. Hence $\alpha(f_\tau) \downarrow 0$ in $\mathcal{L}_n(\mathcal{J}, M)$. For $f \in E$ and $\pi \in Z(M)$ we have

$$\alpha(\pi \cdot f)(T) = T(\pi \cdot f) = \pi Tf = \pi\alpha(f)(T)$$

for all $T \in \mathcal{J}$, so $\alpha(\pi \cdot f) = \pi\alpha(f)$, which shows that α is $Z(M)$-linear. That α is a Riesz homomorphism follows from [13], Lemma 83.19. □

It is clear that α is injective if and only if $\mathcal{J}$ separates the points of E, i.e., for every $0 \neq f \in E$ there exists $T \in \mathcal{J}$ with $Tf \neq 0$ (equivalently, for every $0 < f \in E$ there exists $0 \leq T \in \mathcal{J}$ with $Tf > 0$). If α is injective then, by the above lemma, E and $\alpha(E)$ are isomorphic f-modules over $Z(M)$. The following lemma gives already some more information concerning $\alpha(E)$ as subspace of $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$.

LEMMA 4.8 *If $0 \leq f \in E$ and $0 \leq \Lambda \in \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ such that $0 < \Lambda \leq \tilde{f}$, then there exists $0 \leq g \in E$ such that $0 < \tilde{g} \leq \Lambda$.*

Proof. Since $0 < \Lambda \leq \tilde{f}$, there exists $0 < \varepsilon \in \mathbb{R}$ such that $\Lambda_0 = (\Lambda - \varepsilon\tilde{f})^+ > 0$. Now $\Lambda_0 > 0$ implies that the carrier $\mathcal{C}_{\Lambda_0} \neq \{0\}$, as Λ_0 is order continuous. Take $0 < T_0 \in \mathcal{C}_{\Lambda_0}$. Since $0 < \Lambda_0 \leq \Lambda \leq \tilde{f}$, it follow that

$$T_0 f = \tilde{f}(T_0) \geq \Lambda_0(T_0) > 0.$$

The ideal $\mathcal{C}_{T_0} \oplus \mathcal{N}_{T_0}$ is order dense in E. Since $T_0 \in \mathcal{J}$ in particular implies that T_0 is order continuous, it follows from $T_0 f > 0$ that there exists $f_0 \in \mathcal{C}_{T_0}$ such that $0 < f_0 \leq f$. Since $\mathcal{J}$ itself is Dedekind complete, we have $\mathcal{J} = \mathcal{C}_{\Lambda_0} \oplus \mathcal{N}_{\Lambda_0}$. Take any $0 \leq T \in \mathcal{J}$ and write $T = T_1 + T_2$ with $0 \leq T_1 \in \mathcal{C}_{\Lambda_0}$ and $0 \leq T_2 \in \mathcal{N}_{\Lambda_0}$. Since

$$(\Lambda - \varepsilon \tilde{f})^- \perp (\Lambda - \varepsilon \tilde{f})^+ = \Lambda_0$$

in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$, it follows from Proposition 3.7 that

$$\mathcal{C}_{\Lambda_0} \subseteq \mathcal{N}_{(\Lambda - \varepsilon f)^-}.$$

Consequently, $(\Lambda - \varepsilon \tilde{f})(T_1) = (\Lambda - \varepsilon \tilde{f})^+(T_1) \geq 0$, i.e., $\varepsilon \tilde{f}(T_1) \leq \Lambda(T_1)$. Furthermore, since $0 \leq T_2 \in \mathcal{N}_{\Lambda_0}$ and $0 \leq T_0 \in \mathcal{C}_{\Lambda_0}$, we have $T_2 \wedge T_0 = 0$ in $\mathcal{J} \subseteq \mathcal{L}_n^{Z(M)}(E, M)$. Using Proposition 3.7 once more, it follows that $\mathcal{C}_{T_0} \subseteq \mathcal{N}_{T_2}$. This implies in particular that $T_2 f_0 = 0$, so $T f_0 = T_1 f_0$. Consequently,

$$\begin{aligned} \Lambda(T) &\geq \Lambda(T_1) \geq \varepsilon \tilde{f}(T_1) = \varepsilon T_1 f \geq \\ &\geq \varepsilon T_1 f_0 = \varepsilon T f_0 = (\varepsilon \tilde{f_0})(T). \end{aligned}$$

Hence, putting $g = \varepsilon f_0$, we have $0 \leq \tilde{g}(T) \leq \Lambda(T)$ for all $0 \leq T \in \mathcal{J}$, i.e., $0 \leq \tilde{g} \leq \Lambda$ in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. Finally, it follows from $\tilde{g}(T_0) = \varepsilon T_0 f_0 > 0$ that $0 < \tilde{g} \leq \Lambda$. □

Now we are in the position to prove the main result of the present section.

THEOREM 4.9 *Let M be a Dedekind complete Riesz space, E an f-module over $Z(M)$ and $\mathcal{J}$ an ideal in $\mathcal{L}_n^{Z(M)}(E, M)$ separating the points of E.*
(1) The band generated by $\alpha(E)$ in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ is equal to $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$.
(2) $\alpha(E)$ is an ideal in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ if and only if E is Dedekind complete.
(3) $\alpha(E) = \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ if and only if it follows from $0 \leq f_\tau \uparrow$ in E with $\sup_\tau T f_\tau$ *existing in M for all $0 \leq T \in \mathcal{J}$, that there exists $0 \leq f \in E$ such that $0 \leq f_\tau \uparrow f$.*

Proof. (1) Suppose that $\Lambda \in \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ is such that $\Lambda \perp \alpha(f) = \tilde{f}$ for all $f \in E$. By Proposition 3.7 this implies that $\mathcal{C}_\Lambda \perp \mathcal{C}_{\tilde{f}}$, i.e., $\mathcal{C}_\Lambda \subseteq \mathcal{N}_{\tilde{f}}$ for all $f \in E$. Hence, if $0 \leq T \in \mathcal{C}_\Lambda$ then $Tf = \tilde{f}(T) = 0$ for all $0 \leq f \in E$, and so $T = 0$. This shows that $\mathcal{C}_\Lambda = \{0\}$, hence $\Lambda = 0$.
(2) If $\alpha(E)$ is an ideal in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$, then $\alpha(E)$ is Dedekind complete. Since E and $\alpha(E)$ are Riesz isomorphic, it follows that E is Dedekind complete. Now assume that E is Dedekind complete and suppose that $0 \leq \Lambda \leq \alpha(f)$ in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ for some $0 \leq f \in E$. Define

$$g = \sup\{ 0 \leq h \in E \colon 0 \leq \alpha(h) \leq \Lambda \}.$$

Note that this is well defined, since $0 \leq \alpha(h) \leq \Lambda \leq \alpha(f)$ implies that $0 \leq h \leq f$. Since, by Lemma 4.7, α is an order continuous Riesz homomorphism, it follows

that $0 \leq \alpha(g) \leq \Lambda$. We claim that $\alpha(g) = \Lambda$. Indeed, suppose that $\alpha(g) < \Lambda$. Then $0 < \Lambda - \alpha(g) \leq \alpha(f - g)$. By Lemma 4.8 there exists $0 < g_0 \in E$ such that $0 < \alpha(g_0) \leq \Lambda - \alpha(g)$. Then $0 < \alpha(g + g_0) \leq \Lambda$, and from the definition of g it now follows that $g + g_0 \leq g$, which is a contradiction. Consequently, $\Lambda = \alpha(g) \in \alpha(E)$, and we may conclude that $\alpha(E)$ is an ideal in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$.
(3) First assume that $\alpha(E) = \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$, and suppose that $0 \leq f_\tau \uparrow$ in E such that $\sup_\tau Tf_\tau$ exists for all $0 \leq T \in \mathcal{J}$. Define

$$\Lambda(T) = \sup_\tau Tf_\tau \quad \forall 0 \leq T \in \mathcal{J}.$$

Then $\Lambda\colon \mathcal{J}^+ \to M^+$ is additive and so Λ extends uniquely to a positive linear operator $\Lambda\colon \mathcal{J} \to M$ (see e.g. [13], Lemma 83.1). It follows immediately from the definition of Λ that $0 \leq \tilde{f}_\tau \uparrow \Lambda$ in $\mathcal{L}_b(\mathcal{J}, M)$. Since $\tilde{f}_\tau \in \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ and $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ is a band in $\mathcal{L}_b(\mathcal{J}, M)$, this implies that $0 \leq \Lambda \in \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. By hypothesis there exists $0 \leq f \in E$ such that $\Lambda = \tilde{f} = \alpha(f)$. Since α is a Riesz isomorphism, $\alpha(f_\tau) \uparrow \alpha(f)$ implies that $f_\tau \uparrow f$ in E.
For the converse implication note that the hypothesis on E implies in particular that E is Dedekind complete and so, by (2), $\alpha(E)$ is an ideal in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. Now take $0 \leq \Lambda \in \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. Since, by (1), the band generated by $\alpha(E)$ is equal to $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$, it follows that there exist $0 \leq f_\tau \uparrow$ in E such that $0 \leq \tilde{f}_\tau \uparrow \Lambda$ in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. Then

$$0 \leq Tf_\tau = \tilde{f}_\tau(T) \uparrow \Lambda(T) \quad \forall 0 \leq T \in \mathcal{J},$$

so $\sup_\tau Tf_\tau \in M$ exists for all $0 \leq T \in \mathcal{J}$. By hypothesis there exists $0 \leq f \in E$ such that $f_\tau \uparrow f$. Hence $0 \leq \tilde{f}_\tau \uparrow \tilde{f}$ in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$, which shows that $\Lambda = \tilde{f} = \alpha(f)$. Consequently $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M) = \alpha(E)$, and the proof of the theorem is complete. □

REMARK 4.10 *In case $M = \mathbb{R}$ the above results correspond to the well known results concerning the embedding of a Riesz space into the order continuous bidual (see [13], Section 109, 110). In particular, if $M = \mathbb{R}$ and $\mathcal{J} = \mathcal{L}_n(E, \mathbb{R}) = E_n^\sim$, then (3) of the above theorem corresponds to the perfectness criterion for Riesz spaces ([13], Theorem 110.1). In view of this, if in general E is an f-module over $Z(M)$ satisfying (3) of the above theorem, then we will say that E is M-perfect with respect to $\mathcal{J}$ (or, M-perfect in case $\mathcal{J} = \mathcal{L}_n^{Z(M)}(E, M)$).*

As an example, let L and M be Archimedean Riesz spaces with M Dedekind complete and take $E = \mathcal{L}_b(L, M)$ with its natural $Z(M)$-module structure. Note that $\mathcal{L}_n^{Z(M)}(E, M)$ separates the points of E, as all the mappings $T \mapsto Tf$, with $f \in L$, belong to $\mathcal{L}_n^{Z(M)}(E, M)$. It is not difficult to see that E is M-perfect. This of course is analogous to the order dual $L^\sim$ being perfect for any Riesz space L (see [13], Theorem 110.2).

5. The Construction of Maharam Extensions

In the present section we will be concerned with the construction of Maharam extensions of operators. Let L and M be Archimedean Riesz spaces with M Dedekind complete and let T be a positive operator from L into M. Our aim is to construct a Dedekind complete Riesz space E, containing L as a Riesz subspace, and an order continuous positive Maharam operator $\widehat{T}$ from E into M which extends T. Such an operator $\widehat{T}$ is then called a Maharam extension of the operator T. Actually we will treat a more general situation, where $\mathcal{J}$ is a given ideal of operators in $\mathcal{L}_b(L, M)$, and the problem is to construct a Dedekind complete Riesz space E, containing L as a Riesz subspace, such that all operators $T \in \mathcal{J}$ have Maharam extensions $\widehat{T}: E \to M$ (and the mapping $T \mapsto \widehat{T}$ is a Riesz isomorphism). In view of the results of the previous section (cf. Examples 4.5 (4)), this is equivalent to constructing an f-module E over $Z(M)$, containing L, such that all operators $T \in \mathcal{J}$ have $Z(M)$-linear extensions to E.

Before starting with the construction, we first make some remarks concerning a well known algebraic procedure for so-called extension of scalars. Some aspects of this algebraic construction are reflected in our construction of Maharam extensions.

REMARK 5.1 *Let k be a field, V a vector space over k and A an algebra with unit element e over k. Consider the tensor product $A \otimes V$ of A and V as vector spaces over k. Then $A \otimes V$ is a unital A module with*

$$a \cdot (b \otimes v) = (ab) \otimes v \quad \forall a, b \in A,\ \forall v \in V.$$

The mapping $\chi_0: V \to A \otimes V$, defined by $\chi(v) = e \otimes v$ for all $v \in V$, is a k-linear embedding of V into $A \otimes V$. Now suppose that M is a unital A-module. Since k is embedded in A via the mapping $\lambda \mapsto \lambda e$, M is a k-vector space as well. Let $T: V \to M$ be a k-linear mapping. Then the mapping $(a, v) \mapsto a \cdot Tv$ is k-bilinear and hence there exists a unique k-linear mapping $\bar{T}$ from $A \otimes V$ into M such that

$$\bar{T}(a \otimes v) = a \cdot Tv \quad \forall a \in A,\ \forall v \in V.$$

It is clear that $\bar{T}$ is actually A-linear and that $T = \bar{T}_0\chi_0$ (i.e., $\bar{T}$ extends T). Moreover, $\bar{T}$ is unique with these properties. The A-module $A \otimes V$ is called the extension of V over A.

The construction of Maharam extensions will be more involved than this purely algebraic extension procedure, in particular because the extended space is required to be Dedekind complete and the extended operator is required to be order continuous.

Let L be an Archimedean Riesz space and M a Dedekind complete Riesz space. As before, we consider M as an f-module over $Z(M)$. Then $\mathcal{L}_b(L, M)$ is an f-module over $Z(M)$ as well, with $\pi \cdot T = \pi \circ T$ for all $\pi \in Z(M)$ and

$T \in \mathcal{L}_b(L, M)$. We fix some ideal $\mathcal{J} \subseteq \mathcal{L}_b(L, M)$. With respect to the operations induced by $\mathcal{L}_b(L, M)$ the ideal $\mathcal{J}$ is an f-module over $Z(M)$. For $\pi \in Z(M)$ and $f \in L$ we define the mapping

$$\pi \otimes f : \mathcal{J} \to M$$

by $(\pi \otimes f)(T) = \pi(Tf)$ for all $T \in \mathcal{J}$. It is clear that $\pi \otimes f \in \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. Now we define the mapping

$$\Psi : Z(M) \times L \to \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$$

by $\Psi(\pi, f) = \pi \otimes f$.

LEMMA 5.2 *Ψ is a Riesz bimorphism, i.e., Ψ is bilinear and $|\Psi(\pi, f)| = \Psi(|\pi|, |f|)$ for all $\pi \in Z(M)$ and $f \in L$.*

Proof. The bilinearity is obvious. Take $\pi \in Z(M)$, $f \in L$ and $0 \leq T \in \mathcal{J}$. Using [13], Theorem 83.9 we find that

$$\begin{aligned} |\pi \otimes f|(T) &= \sup\{\, |(\pi \otimes f)(S)| : S \in \mathcal{J},\ |S| \leq T \,\} = \\ &= \sup\{\, |\pi(Sf)| : S \in \mathcal{J},\ |S| \leq T \,\} = \\ &= \sup\{\, |\pi|(|Sf|) : S \in \mathcal{J},\ |S| \leq T \,\} = \\ &= |\pi| \sup\{\, |Sf| : S \in \mathcal{L}_b(L, M),\ |S| \leq T \,\} = \\ &= |\pi|(T|f|) = (|\pi| \otimes |f|)(T). \end{aligned}$$

Hence $|\pi \otimes f| = |\pi| \otimes |f|$ in $\mathcal{L}_b^{Z(M)}(\mathcal{J}, M)$. □

We denote by $Z(M) \otimes_{\mathcal{J}} L$ the linear subspace of $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ generated by the operators of the form $\pi \otimes f$, i.e.,

$$Z(M) \otimes_{\mathcal{J}} L = \left\{ \sum_{i=1}^{n} \pi_i \otimes f_i : \pi_i \in Z(M),\ f_i \in L,\ i = 1, \cdots, n;\ n \in \mathbb{N} \right\}.$$

Furthermore, $Z(M) \widetilde{\otimes}_{\mathcal{J}} L$ will denote the ideal generated by $Z(M) \otimes_{\mathcal{J}} L$ in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. The mapping

$$\chi_0 : L \to \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$$

is defined by $\chi_0 f = I \otimes f$ for all $f \in L$. It follows from Lemma 5.2 that χ_0 is a Riesz homomorphism and it is obvious that χ_0 maps L into $Z(M) \widetilde{\otimes}_{\mathcal{J}} L$. So we may consider $\chi_0 : L \to Z(M) \widetilde{\otimes}_{\mathcal{J}} L$ as well.

LEMMA 5.3 *$Z(M) \widetilde{\otimes}_{\mathcal{J}} L$ is equal to the ideal generated by $\chi_0(L)$ in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$, i.e.,*

$$Z(M) \widetilde{\otimes}_{\mathcal{J}} L = \left\{ \theta \in \mathcal{L}_n^{Z(M)}(\mathcal{J}, M) : |\theta| \leq I \otimes u \text{ for some } 0 \leq u \in L \right\}.$$

Proof. Take $\theta \in Z(M)\widetilde{\otimes}_{\mathcal{J}} L$. Then there exist $\pi_1, \cdots, \pi_n \in Z(M)$ and $f_1, \cdots, f_n \in L$ such that

$$|\theta| \leq |\sum_{i=1}^{n} \pi_i \otimes f_i|.$$

Define $u = \sum_{i=1}^{n} |f_i|$ and $\pi = \sum_{i=1}^{n} |\pi_i|$. Since $0 \leq \pi \in Z(M)$ there exists $0 \leq k \in \mathbb{R}$ such that $0 \leq \pi \leq kI$.
Hence,

$$\begin{aligned} |\theta| \ &\leq \ \sum_{i=1}^{n} |\pi_i \otimes f_i| = \sum_{i=1}^{n} |\pi_i| \otimes |f_i| \leq \\ &\leq \ \pi \otimes u \leq (kI) \otimes u = I \otimes (ku) = \chi_0(ku), \end{aligned}$$

which suffices to prove the lemma. □

Now take any ideal E in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ such that

$$Z(M)\widetilde{\otimes}_{\mathcal{J}} L \subseteq E \subseteq \mathcal{L}_n^{Z(M)}(\mathcal{J}, M).$$

Note that E is Dedekind complete and that E is an f-submodule of $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. Furthermore observe that E separates the points of $\mathcal{J}$. Indeed, if $T \in \mathcal{J}$ is such that $\theta(T) = 0$ for all $\theta \in E$, then in particular $Tf = (I \otimes f)(T) = 0$ for all $f \in L$, so $T = 0$.

Define the mapping

$$\alpha_E : \mathcal{J} \to \mathcal{L}_n^{Z(M)}(E, M)$$

by $\alpha_E(T) = \widehat{T}$, where $\widehat{T}(\theta) = \theta(T)$ for all $\theta \in E$. It follows from the results in Section 4, in particular Theorem 4.9, that α_E is a Riesz isomorphism and that $\alpha_E(\mathcal{J})$ is an order dense ideal in $\mathcal{L}_n^{Z(M)}(E, M)$. Note that $\widehat{T}(\pi \otimes f) = \pi(Tf)$ for all $\pi \in Z(M)$ and $f \in L$. In particular $\widehat{T}(\chi_0 f) = \widehat{T}(I \otimes f) = Tf$ for all $f \in L$, i.e., $T = \widehat{T}_0 \chi_0$. So for every $T \in \mathcal{J}$ we have the commutative diagram

$$\begin{tikzcd} L \arrow[rr, "T"] \arrow[dr, "\chi_0"'] & & M \\ & E \arrow[ur, "\widehat{T}"'] & \end{tikzcd}$$

where χ_0 is a Riesz homomorphism and $\widehat{T}$ is a Maharam operator. This already proves some of the main parts of the following theorem.

THEOREM 5.4 *Let L, M, $\mathcal{J} \subseteq \mathcal{L}_b(L, M)$ and $E \subseteq \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ be as above. Then:*

(i) E is a Dedekind complete f-module over $Z(M)$;

(ii) $\chi_0\colon L \to E$ is a Riesz homomorphism such $\chi_0(L)$ is order dense in E (i.e., $\chi_0(L)^d = \{0\}$);

(iii) for every $T \in \mathcal{J}$ the operator $\widehat{T} = \alpha_E(T)$ is the unique operator in $\mathcal{L}_n^{Z(M)}(E, M)$ such that $T = \widehat{T_0}\chi_0$ and α_E is a Riesz isomorphism from $\mathcal{J}$ onto an order dense ideal in $\mathcal{L}_n^{Z(M)}(E, M)$;

(iv) $\{\widehat{T}\colon 0 \le T \in \mathcal{J}\}$ separates the points of E.

Moreover, if $\mathcal{J}$ is a band in $\mathcal{L}_b(L, M)$, then α_E is a Riesz isomorphism from $\mathcal{J}$ onto $\mathcal{L}_n^{Z(M)}(E, M)$.

Proof. (i) This has been observed above.

(ii) Suppose $\theta \in E$ such that $\theta \in \chi_0(L)^d$. Then $\theta \perp \chi_0 f$ in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ for all $f \in L$. It follows from Proposition 3.7 that $\mathcal{C}_\theta \perp \mathcal{C}_{\chi_0 f}$, i.e., $\mathcal{C}_\theta \subseteq \mathcal{N}_{\chi_0 f}$in $\mathcal{J}$ for all $f \in L$. Hence, if $T \in \mathcal{C}_\theta$ then $Tf = (\chi_0 f)(T) = 0$ for all $f \in L$. This shows that $\mathcal{C}_\theta = \{0\}$ and so $\theta = 0$.

(iii) Only the uniqueness statement has still to be proved. For this purpose define the operator

$$R\colon \mathcal{L}_n^{Z(M)}(E, M) \to \mathcal{L}_b(L, M)$$

by $R(\Lambda) = \Lambda \circ \chi_0$ for all $\Lambda \in \mathcal{L}_n^{Z(M)}(E, M)$, which is clearly positive and order continuous. Moreover, $R(\widehat{T}) = T$ for all $T \in \mathcal{J}$. We claim that R is a Riesz homomorphism. Indeed, suppose that $\Lambda_1 \wedge \Lambda_2 = 0$ in $\mathcal{L}_n^{Z(M)}(E, M)$. Since $\alpha_E(\mathcal{J})$ is order dense in $\mathcal{L}_n^{Z(M)}(E, M)$ there exist $0 \le T_\tau \uparrow$ and $0 \le S_\sigma \uparrow$ in $\mathcal{J}$ such that $0 \le \widehat{T_\tau} \uparrow \Lambda_1$ and $0 \le \widehat{S_\sigma} \uparrow \Lambda_2$ in $\mathcal{L}_n^{Z(M)}(E, M)$. Now

$$0 \le \alpha_E(T_\tau \wedge S_\sigma) = \widehat{T_\tau} \wedge \widehat{S_\sigma} \le \Lambda_1 \wedge \Lambda_2 = 0,$$

so $\alpha_E(T_\tau \wedge S_\sigma) = 0$ and since α_E is injective this implies that $T_\tau \wedge S_\sigma = 0$ for all τ, σ. By the order continuity of R we have $0 \le T_\tau \uparrow R(\Lambda_1)$ and $0 \le S_\sigma \uparrow R(\Lambda_2)$ in $\mathcal{L}_b(L, M)$. Consequently $R(\Lambda_1) \wedge R(\Lambda_2) = 0$, which proves the claim.
Now we show that R is injective. Suppose $\Lambda \in \mathcal{L}_n^{Z(M)}(E, M)$ such that $R(\Lambda) = 0$. Since R is a Riesz homomorphism this implies that $R(|\Lambda|) = 0$. Hence $|\Lambda|(\chi_0 f) = 0$ for all $0 \le f \in L$. From the order density of $\chi_0(L)$ in E it follows that $\Lambda = 0$. Now it is obvious that if $T \in \mathcal{J}$ and $\Lambda \in \mathcal{L}_n^{Z(M)}(E, M)$ such that $T = \widehat{T} \circ \chi_0 = \Lambda \circ \chi_0$, then $\Lambda = \widehat{T}$.

(iv) This follows immediately from the definitions.

To prove the last statement of the theorem, now assume that $\mathcal{J}$ is a band in $\mathcal{L}_b(L, M)$. Suppose that $0 \le T_\tau \uparrow$ in $\mathcal{J}$ such that $\sup_\tau \theta(T_\tau) \in M$ exists for all $0 \le \theta \in E$. By taking $\theta = I \otimes f$, $0 \le f \in L$, we see that $\sup_\tau T_\tau f \in M$ exists for all $0 \le f \in L$. Define $Tf = \sup_\tau T_\tau f$ for all $0 \le f \in L$ and extend T to a positive operator $0 \le T \in \mathcal{L}_b(L, M)$. Then $0 \le T_\tau \uparrow T$ in $\mathcal{L}_b(L, M)$, and since $\mathcal{J}$ is a band we conclude that $0 \le T_\tau \uparrow T \in \mathcal{J}$. By Theorem 4.9 (iii) it follows that α_E is surjective. □

Actually the properties of ideals $E \subseteq \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ obtained in Theorem 5.4 characterize such ideals. This is the contents of the next proposition.

PROPOSITION 5.5 *Let L and M be Archimedean Riesz spaces with M Dedekind complete and let $\mathcal{J} \subseteq \mathcal{L}_b(L, M)$ be an ideal. Let the pair (F, χ_F) be such that:*

(i) F is a Dedekind complete f-module over $Z(M)$ and $\mathcal{L}_n^{Z(M)}(F, M)$ separates the points of F;

(ii) χ_F is a Riesz homomorphism from L into F

(iii) for every $T \in \mathcal{J}$ there exists a unique $\widetilde{T} \in \mathcal{L}_n^{Z(M)}(F, M)$ such that $T = \widetilde{T} \circ \chi_F$ and the mapping ω_F defined by $\omega_F(T) = \widetilde{T}$ is a Riesz isomorphism from $\mathcal{J}$ onto an order dense ideal in $\mathcal{L}_n^{Z(M)}(F, M)$.

Then there exists a unique ideal E in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ such that $\chi_0(L) \subseteq E$ and a unique $Z(M)$-linear Riesz isomorphism γ from F onto E such that $\gamma \circ \chi_F = \chi_0$.

Proof. First note that ω_F is order continuous, as ω_F is an injective Riesz homomorphism and $\omega_F(\mathcal{J})$ is an ideal. For $\mu \in \mathcal{L}_n^{Z(M)}\big(\mathcal{L}_n^{Z(M)}(F, M), M\big)$ we define $\omega_F^*\mu : \mathcal{J} \to M$ by $(\omega_F^*\mu)(T) = \mu(\omega_F T)$ for all $T \in \mathcal{J}$. Since ω_F is order continuous and $Z(M)$-linear, it follows that $\omega_F^*\mu \in \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. Hence this defines a linear mapping

$$\omega_F^* : \mathcal{L}_n^{Z(M)}\big(\mathcal{L}_n^{Z(M)}(F, M), M\big) \to \mathcal{L}_n^{Z(M)}(\mathcal{J}, M).$$

Using that ω_F is a Riesz homomorphism and that $\omega_F(\mathcal{J})$ is an order dense ideal in $\mathcal{L}_n^{Z(M)}(F, M)$ it is easy to see that ω_F^* is a Riesz homomorphism which is Maharam (cf, [3], Theorem 7.4), injective and clearly $Z(M)$-linear.

Since F is Dedekind complete and $\mathcal{L}_n^{Z(M)}(F, M)$ separates the points of F, it follows from Lemma 4.7 and Theorem 4.9 that the mapping

$$j_F : F \to \mathcal{L}_n^{Z(M)}\big(\mathcal{L}_n^{Z(M)}(F, M), M\big),$$

defined by $(j_F h)(\Lambda) = \Lambda(h)$ for all $\Lambda \in \mathcal{L}_n^{Z(M)}(F, M)$, $h \in F$, is a $Z(M)$-linear Riesz isomorphism onto an order dense ideal. Now define $E = (\omega_F^* \circ j_F)(F)$. From the above it follows that E is an ideal in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ and that $\gamma = \omega_F^* \circ j_F$ is a $Z(M)$-linear Riesz isomorphism from F onto E. That $\gamma \circ \chi_F = \chi_0$ follows easily from a repeated application of the appropriate definitions. From this it also follows that $\chi_0(L) \subseteq E$.

It remains to show that E and γ are unique. This will follow at once from the following claim. Suppose that E is an ideal in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ with $\chi_0(L) \subseteq E$ and that $\beta : E \to \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ is $Z(M)$-linear and order continuous such that $\beta \circ \chi_0 = \chi_0$. Then $\beta(\theta) = \theta$ for all $\theta \in E$. To prove this claim we denote $\widehat{T} = \alpha_E(T) \in \mathcal{L}_n^{Z(M)}(E, M)$ for $T \in \mathcal{J}$. Now define $\Lambda_T \in \mathcal{L}_n^{Z(M)}(E, M)$ by $\Lambda_T(\theta) = \widehat{T}(\beta\theta)$ for all $\theta \in E$. Then $\Lambda_T(\chi_0 f) = \widehat{T}(\beta\chi_0 f) = \widehat{T}(\chi_0 f) = Tf$ for all $f \in L$. From the uniqueness statement in Theorem 5.4 (ii) it follows that $\Lambda_T = \widehat{T}$. This shows that $(\beta\theta)(T) = \theta(T)$ for all $T \in \mathcal{J}$ and all $\theta \in E$. Hence $\beta(\theta) = \theta$ for all $\theta \in E$. $\square$

Motivated by the above results we come to the following definition. As before, L is an Archimedean Riesz space, M a Dedekind complete Riesz space and $\mathcal{J} \subseteq \mathcal{L}_b(L, M)$ an ideal.

DEFINITION 5.6 *A Maharam extension space for $\mathcal{J}$ is a pair (F, χ_F) such that:*

(i) F is a Dedekind complete f-module over $Z(M)$ and $\chi_F : L \to F$ is a Riesz homomorphism;

(ii) for every $0 \leq T \in \mathcal{J}$ there exists a unique $0 \leq \widetilde{T} \in \mathcal{L}_n^{Z(M)}(F, M)$ such that $T = \widetilde{T} \circ \chi_F$;

(iii) $\{\widetilde{T} : 0 \leq T \in \mathcal{J}\}$ separates the points of F.

Moreover, (F, χ_F) is called minimal if the ideal generated by $\chi_F(L)$ is equal to F; it is called maximal if $0 \leq h_\tau \uparrow$ in F with $\sup_\tau \widetilde{T} h_\tau \in M$ exists for all $0 \leq T \in \mathcal{J}$, implies that $0 \leq h_\tau \uparrow h \in F$.

It follows from Theorem 5.4 that (E, χ_0) is a Maharam extension space for any ideal E in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ with $\chi_0(L) \subseteq E$. Moreover, $Z(M)\widetilde{\otimes}_{\mathcal{J}} M$ is a minimal Maharam extension space, and it is easy to see that $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ is maximal. Indeed, suppose $0 \leq \theta_\tau \uparrow$ in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ such that $\sup_\tau \widetilde{T}(\theta_\tau) \in M$ exists, i.e., $\sup_\tau \theta_\tau(T) \in M$ exists for all $0 \leq T \in \mathcal{J}$. Define $\theta(T) = \sup_\tau \theta_\tau(T)$ for all $0 \leq T \in \mathcal{J}$. This gives $0 \leq \theta \in \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ and $0 \leq \theta_\tau \uparrow \theta$.

Now we will show that, up to isomorphism, these ideals in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ are the only Maharam extension spaces for $\mathcal{J}$.

PROPOSITION 5.7 *Let (F, χ_F) be a Maharam extension space for the ideal $\mathcal{J} \subseteq \mathcal{L}_b(L, M)$. Define*

$$R : \mathcal{L}_n^{Z(M)}(F, M) \to \mathcal{L}_b(L, M)$$

by $R(\Lambda) = \Lambda \circ \chi_F$ for all $\Lambda \in \mathcal{L}_n^{Z(M)}(F, M)$.

(1) R is a Riesz isomorphism (into) and $\{\Lambda \in \mathcal{L}_n^{Z(M)}(F, M) : R(\Lambda) \in \mathcal{J}\}$ is an order dense ideal in $\mathcal{L}_n^{Z(M)}(F, M)$.

(2) For every $T \in \mathcal{J}$ there exists a unique $\widetilde{T} \in \mathcal{L}_n^{Z(M)}(F, M)$ such that $T = \widetilde{T} \circ \chi_F$ and the mapping $T \mapsto \widetilde{T}$ is a Riesz isomorphism from $\mathcal{J}$ onto some order dense ideal in $\mathcal{L}_n^{Z(M)}(F, M)$.

The proof of the above proposition is based on a general argument which we formulate for convenience as a separate lemma.

LEMMA 5.8 *Let X and Y be Archimedean Riesz spaces, $J \subseteq Y$ an ideal and suppose that $R : X \to Y$ is a positive order continuous operator such that:*

(i) for every $0 \leq y \in J$ there exists a unique $0 \leq \widetilde{y} \in X$ such that $R(\widetilde{y}) = y$;

(ii) $\{\widetilde{y} : 0 \leq y \in J\}^d = \{0\}$.

Then R is a Riesz isomorphism (into), $A = R^{-1}(J)$ is an order dense ideal in X and $R(A) = J$.

Proof. It follows from (i) that $(y_1+y_2)^{\sim} = \widetilde{y_1}+\widetilde{y_2}$ for all $0 \leq y_1, y_2 \in J$. Hence there exists a unique positive linear operator $S\colon J \to X$ such that $Sy = \widetilde{y}$ for all $0 \leq y \in J$. It is clear that $RSy = y$ for all $y \in J$. Moreover it follows from (i) that if $0 \leq x \in X$ such that $Rx \in J$, then $S(Rx) = x$. Now suppose that $x \in X$ is such that $|x| \leq |Sy|$ for some $y \in J$. Then $0 \leq x^+ \leq S|y|$, so $0 \leq R(x^+) \leq |y|$ and hence $R(x^+) \in J$. As observed above, this implies that $S(Rx^+) = x^+$. Similarly, $S(Rx^-) = x^-$ and so $S(Rx) = x$. This shows that $A = S(J)$ is an ideal in X and that $SRx = x$ for all $x \in A$. Let $R_0\colon A \to J$ be the restriction of R to A. It is now clear that $S\colon J \to A$ is the inverse of R_0. Consequently, R_0 is a Riesz isomorphism. From (ii) it follows that the ideal A is order dense. Using that R is order continuous, it now follows that R is a Riesz homomorphism. The kernel of R is a band in X and $\ker(R) \cap A = \{0\}$. Hence $\ker(R) = \{0\}$, and so R is a Riesz isomorphism. Finally, if $x \in X$ such that $Rx \in J$, then $R|x| \in J$ so $|x| = SR|x| \in A$, which shows that $A = R^{-1}(J)$. $\square$

Proof of Proposition 5.7. By the definition of a Maharam extension space the collection $\{\widetilde{T}: 0 \leq T \in \mathcal{J}\}$ separates the points of F. Suppose that $0 \leq \Lambda \in \mathcal{L}_n^{Z(M)}(F, M)$ is such that $\Lambda \wedge \widetilde{T} = 0$ for all $0 \leq T \in \mathcal{J}$. It follows from Proposition 3.7 that $\mathcal{C}_\Lambda \perp \mathcal{C}_{\widetilde{T}}$, i.e., $\mathcal{C}_\Lambda \subseteq \mathcal{N}_{\widetilde{T}}$ for all $0 \leq T \in \mathcal{J}$. This implies that $\mathcal{C}_\Lambda = \{0\}$ and hence $\Lambda = 0$. Consequently, $\{\widehat{T}: 0 \leq T \in \mathcal{J}\}^d = \{0\}$ in $\mathcal{L}_n^{Z(M)}(F, M)$. Now the result of the proposition follows immediately from the above lemma. $\square$

THEOREM 5.9 *Let $\mathcal{J} \subseteq \mathcal{L}_b(L, M)$ be an ideal as above and let (F, χ_F) be a Maharam extension space for $\mathcal{J}$. Then there exists a unique ideal E in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ with $\chi_0(L) \subseteq E$ and a unique $Z(M)$-linear Riesz isomorphism γ from F onto E such that $\gamma \circ \chi_F = \chi_0$. Moreover, (F, χ_F) is minimal if and only if $E = Z(M)\widetilde{\otimes}_{\mathcal{J}} L$ and (F, χ_F) is maximal if and only if $E = \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$.*

Proof. First observe that it follows in particular from (iii) in Definition 5.6 that $\mathcal{L}_n^{Z(M)}(F, M)$ separates the points of F. Furthermore, by Proposition 5.7 (2) there exists a Riesz isomorphism ω_F from $\mathcal{J}$ onto an order dense ideal in $\mathcal{L}_n^{Z(M)}(F, M)$ such that $\widetilde{T} = \omega_F(T)$ is the unique operator in $\mathcal{L}_n^{Z(M)}(F, M)$ satisfying $T = \widetilde{T} \circ \chi_F$. The first statement of the theorem follows therefore immediately from Proposition 5.5. It is obvious that F is minimal if and only if $E = Z(M)\widetilde{\otimes}_{\mathcal{J}} L$. Denoting, as before, by $\widehat{T}$ the unique operator in $\mathcal{L}_n^{Z(M)}(E, M)$ such that $T = \widehat{T} \circ \chi_0$, it is clear that $\widetilde{T} = \widehat{T} \circ \gamma$ for all $T \in \mathcal{J}$. This implies that (F, χ_F) is maximal if and only if (E, χ_0) is maximal. Therefore, it remains to show that (E, χ_0) is maximal if and only if $E = \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. We have already observed, after Definition 5.6, that $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ is maximal. Now assume that $(E, \chi_0$ is maximal. Take $0 \leq \theta \in \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. Since E is an order dense ideal in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ there exist $\theta_\tau \in E$ such that $0 \leq \theta_\tau \uparrow \theta$ in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. Then $\theta_\tau(T) \uparrow \theta(T)$, i.e., $\widehat{T}(\theta_\tau) \uparrow \theta(T)$ in M for all $0 \leq T \in \mathcal{J}$. Since E is maximal and $\sup_\tau \widehat{T}(\theta_\tau) \in M$ exists for all $0 \leq T \in \mathcal{J}$, there exists $\theta_0 \in E$ such that $0 \leq \theta_\tau \uparrow \theta_0$ in E, and hence $0 \leq \theta_\tau \uparrow \theta_0$ in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ as E is an ideal. We may conclude that $\theta = \theta_0 \in E$. This shows that $E = \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. $\square$

As the above theorem shows, any Maharam extension space can be considered canonically as an ideal in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$. Moreover any minimal Maharam extension space in canonically isomorphic to $Z(M)\widetilde{\otimes}_{\mathcal{J}}L$, which we call *the* minimal Maharam extension space of $\mathcal{J}$. Similarly *the* maximal Maharam extension space of $\mathcal{J}$ is uniquely determined, up to isomorphism. We will denote this maximal extension space by $Z(M)\widehat{\otimes}_{\mathcal{J}}L$, and we take $Z(M)\widehat{\otimes}_{\mathcal{J}}L = \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$.

REMARK 5.10 *(1) Let (E, χ) be a Maharam extension space for the ideal $\mathcal{J} \subseteq \mathcal{L}_b(L, M)$. Then for every $0 \leq T \in \mathcal{J}$ there exists a unique $0 \leq \widehat{T} \in \mathcal{L}_n^{Z(M)}(E, M)$ such that $T = \widehat{T} \circ \chi$. In particular, $\widehat{T}$ is a Maharam operator. We note that, in general, there exist Maharam operators $0 \leq S \in \mathcal{L}_n(E, M)$ such that $T = S \circ \chi$, but $S \neq \widehat{T}$ (see Example 6.B). Therefore, for the uniqueness of $\widehat{T}$, it is essential to require $Z(M)$-linearity with respect to some fixed $Z(M)$-module structure of the space E.*
(2) If (E, χ) is a Maharam extension space for the ideal $\mathcal{J} \subseteq \mathcal{L}_b(L, M)$, then χ need not be injective. It is not difficult to show that χ is injective if and only if $\mathcal{J}$ separates the points of L. Therefore, if $\mathcal{J}$ separates the points of L, then χ is a Riesz isomorphism of L into E, so we may consider L as a Riesz subspace of E. For each $T \in \mathcal{J}$ the corresponding operator $\widehat{T} \in \mathcal{L}_n^{Z(M)}(E, M)$ is then an extension of T. Therefore we will call $\widehat{T}$ the Maharam extension of T with respect to (E, χ).

If (E, χ) is a Maharam extension space for the ideal $\mathcal{J}$ in $\mathcal{L}_b(L, M)$, then χ is in general not order continuous. In this connection we have the following simple result.

PROPOSITION 5.11 *The Riesz homomorphism χ is order continuous if and only if $\mathcal{J} \subseteq \mathcal{L}_n(L, M)$.*

Proof. Suppose that $\mathcal{J} \subseteq \mathcal{L}_n(L, M)$ and take $u_\tau \downarrow 0$ in L. Let $\theta = \inf_\tau \chi(u_\tau)$ in E and assume that $\theta > 0$. For $T \in \mathcal{J}$ we denote by $\widehat{T}$ the corresponding operator in $\mathcal{L}_n^{Z(M)}(E, M)$. Since $\{\widehat{T} : 0 \leq T \in \mathcal{J}\}$ separates the points of E, there exists $0 \leq T \in \mathcal{J}$ such that $\widehat{T}\theta > 0$. Then $\widehat{T}(\chi u_\tau) = T u_\tau \downarrow 0$ in M, but on the other hand $\widehat{T}(\chi u_\tau) \geq \widehat{T}\theta > 0$, which is a contradiction. This shows that χ is order continuous.

Since $T = \widehat{T} \circ \chi$ for all $T \in \mathcal{J}$, the converse implication is obvious. □

Let (E, χ) be a Maharam extension space for the ideal $\mathcal{J} \subseteq \mathcal{L}_b(L, M)$ and let $\widehat{T} \in \mathcal{L}_n^{Z(M)}(E, M)$ denote the operator corresponding to $T \in \mathcal{J}$. The mapping $\alpha : T \mapsto \widehat{T}$ is a Riesz isomorphism from $\mathcal{J}$ into $\mathcal{L}_n^{Z(M)}(E, M)$. If $\mathcal{J}$ is a band in $\mathcal{L}_b(L, M)$, then it follows from Theorem 5.4 that α is surjective. If E is the minimal Maharam extension space, then actually the converse of this statement holds, as is shown in the next proposition.

PROPOSITION 5.12 *Let $E = Z(M)\widetilde{\otimes}_{\mathcal{J}}L$, the minimal Maharam extension space of the ideal $\mathcal{J} \subseteq \mathcal{L}_b(L, M)$ and let $\mathcal{B}$ denote the band generated by $\mathcal{J}$ in $\mathcal{L}_b(L, M)$.*

(1) E is the minimal Maharam extension space of $\mathcal{B}$, i.e., $Z(M)\widetilde{\otimes}_{\mathcal{J}}L = Z(M)\widetilde{\otimes}_{\mathcal{B}}L$.

(2) The Riesz isomorphism $T \mapsto \widehat{T}$ from $\mathcal{J}$ into $\mathcal{L}_n^{Z(M)}(E, M)$ is surjective if and only if $\mathcal{J} = \mathcal{B}$.

Proof. (1) Consider the mapping

$$R\colon \mathcal{L}_n^{Z(M)}(E, M) \to \mathcal{L}_b(L, M)$$

defined by $R(\Lambda) = \Lambda \circ \chi$ for all $\Lambda \in \mathcal{L}_n^{Z(M)}(E, M)$. Then R is an injective Riesz homomorphism (see the proof of Theorem 5.4 (iii), or Proposition 5.7 (1)). It clearly suffices to show that for every $0 \leq T \in \mathcal{B}$ there exists $0 \leq \Lambda \in \mathcal{L}_n^{Z(M)}(E, M)$ such that $R(\Lambda) = T$. To this end, let $0 \leq T \in \mathcal{B}$ be given and take $T_\tau \in \mathcal{J}$ such that $0 \leq T_\tau \uparrow T$ in $\mathcal{L}_b(L, M)$. If $0 \leq \theta \in E$ then $0 \leq \theta \leq I \otimes u$ for some $0 \leq u \in L$, so $\widehat{T_\tau}(\theta) \leq \widehat{T_\tau}(I \otimes u) = T_\tau u \leq Tu$ for all τ. Hence, $\Lambda(\theta) = \sup_\tau \widehat{T_\tau}(\theta) \in M$ exists for all $0 \leq \theta \in M$, and this defines $\Lambda \in \mathcal{L}_n^{Z(M)}(E, M)$ such that $\widehat{T_\tau} \uparrow \Lambda$. Since R is order continuous it follow now that $T_\tau = R(\widehat{T_\tau}) \uparrow R(\Lambda)$ in $\mathcal{L}_b(L, M)$, hence $R(\Lambda) = T$.

(2) It follows from (1) that for every $T \in \mathcal{B}$ there exists a unique $\widehat{T} \in \mathcal{L}_n^{Z(M)}(E, M)$ such that $T = \widehat{T} \circ \chi$, and the mapping $T \mapsto \widehat{T}$ is a Riesz isomorphism from $\mathcal{B}$ onto $\mathcal{L}_n^{Z(M)}(E, M)$. This immediately implies (2). □

Next we present a characterization of Maharam extension spaces of ideals generated by subsets D of $\mathcal{L}_b(L, M)^+$. This result is sometimes useful for identifying Maharam extension spaces in concrete situations.

PROPOSITION 5.13 *As before, let L and M be Archimedean Riesz spaces with M Dedekind complete and let D be a subset of $\mathcal{L}_b(L, M)^+$. Suppose that E is a Dedekind complete f-module over $Z(M)$ and $\chi : L \to E$ a Riesz homomorphism such that:*

(1) for every $T \in D$ there exists a unique $0 \leq \widehat{T} \in \mathcal{L}_n^{Z(M)}(E, M)$ with $T = \widehat{T} \circ \chi$;

(2) $\{\widehat{T} : T \in D\}$ separates the points of E.

Then (E, χ) is a Maharam extension space for the ideal $\mathcal{J}$ generated by D in $\mathcal{L}_b(L, M)$.

Proof. As before we consider the mapping R from $\mathcal{L}_n^{Z(M)}(E, M)$ into $\mathcal{L}_b(L, M)$ given by $R(\Lambda) = \Lambda \circ \chi$. Then R is a positive Maharam operator, as χ is a Riesz homomorphism (see, e.g., [3], Theorem 7.4). Suppose that $0 \leq T_1, T_2 \in \mathcal{L}_b(L, M)$ are such that there exist unique $0 \leq \widehat{T_1}, \widehat{T_2} \in \mathcal{L}_n^{Z(M)}(E, M)$ with $R(\widehat{T_j}) = T_j$ ($j = 1, 2$). We claim that $T_1 + T_2$ has the same property. Indeed, suppose that $0 \leq \Lambda \in \mathcal{L}_n^{Z(M)}(E, M)$ such that $R(\Lambda) = T_1 + T_2$. Then $0 \leq T_1 \leq R(\Lambda)$ implies that $T_1 = R(\Lambda_1)$ for some $0 \leq \Lambda_1 \leq \Lambda$. From the hypothesis on T_1 it follows that $\Lambda_1 = \widehat{T_1}$, hence $0 \leq \widehat{T_1} \leq \Lambda$. Now $\Lambda - \widehat{T_1} \geq 0$ and $R(\Lambda - T_1) = T_2$, so $\Lambda - \widehat{T_1} = \widehat{T_2}$, which proves the claim.

Consequently, we may assume without loss of generality that D is upwards directed. Now take $0 \leq S \in \mathcal{J}$. Then there exist $T \in D$ and $n \in \mathbb{N}$ such that $0 \leq$

$S \leq nT = R(n\widehat{T})$. Hence there exists $\widehat{S} \in \mathcal{L}_n^{Z(M)}(E, M)$ such that $0 \leq \widehat{S} \leq n\widehat{T}$ and $R(\widehat{S}) = S$. Suppose that $0 \leq \Lambda \in \mathcal{L}_n^{Z(M)}(E, M)$ also satisfies $R(\Lambda) = S$. Then $n\widehat{T} + (\Lambda - \widehat{S}) \geq 0$ and $R\big[n\widehat{T} + (\Lambda - \widehat{S})\big] = R(n\widehat{T})$. From (1) it follows that $\Lambda = \widehat{S}$. This suffices to prove the proposition. □

COROLLARY 5.14 *Let* $0 \leq T_0 \in \mathcal{L}_b(L, M)$. *Suppose that* E *is a Dedekind complete* f*-module over* $Z(M)$ *and* $\chi : L \to E$ *a Riesz homomorphism satisfying*
(1) there exists a unique $0 \leq \widehat{T}_0 \in \mathcal{L}_n^{Z(M)}(E, M)$ *such that* $T_0 = \widehat{T}_0 \circ \chi$*;*
(2) $\widehat{T}_0 h > 0$ *for all* $0 < h \in E$.
Then (E, χ) *is a Maharam extension space for the ideal* $\mathcal{J}_{T_0}$ *generated by* T_0 *in* $\mathcal{L}_b(L, M)$.

Moreover, if we assume that E *is minimal (i.e., the ideal generated by* $\chi(L)$ *is equal to* E*), then* (E, χ) *is the minimal extension space for* $\{T_0\}^{dd}$ *as well, so for each* $T \in \{T_0\}^{dd}$ *there exists a unique* $\widehat{T} \in \mathcal{L}_n^{Z(M)}(E, M)$ *with* $T = \widehat{T} \circ \chi$ *and the mapping* $T \mapsto \widehat{T}$ *is a Riesz isomorphism from* $\{T_0\}^{dd}$ *onto* $\mathcal{L}_n^{Z(M)}(E, M)$.

6. Examples

In this section we will discuss some examples of Maharam extensions, illustrating the results in the previous section.

Example A

In this first example we consider finite dimensional spaces. Let $L = \mathbb{R}^m$ and $M = \mathbb{R}^n$, both with coordinatewise ordering. By $\{e_1, \cdots, e_m\}$ and $\{f_1, \cdots, f_n\}$ we denote the standard bases in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Every linear operator T from L into M can be represented by an $n \times m$ matrix (τ_{ij}) with respect to these basis. The center $Z(M)$ corresponds to all $n \times n$ diagonal matrices and can be identified with $\mathbb{R}^n$. If $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ then the corresponding operator $\pi_\lambda \in Z(M)$ is given by $\pi_\lambda(y) = (\lambda_1\eta_1, \cdots, \lambda_n\eta_n)$ for all $y = (\eta_1, \cdots, \eta_n) \in M$.

We take $\mathcal{J} = \mathcal{L}_b(L, M) = \mathcal{L}(L, M)$. Let $E = \mathbb{R}^{mn}$ with coordinatewise ordering and standard basis $\{h_1, \cdots, h_{mn}\}$. For $z = (\zeta_1, \cdots, \zeta_{mn}) \in E$ and $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n = Z(M)$ we define

$$\lambda \cdot z = (\lambda_1\zeta_1, \cdots, \lambda_1\zeta_m, \lambda_2\zeta_{m+1}, \cdots, \lambda_2\zeta_{2m}, \cdots, \lambda_n\zeta_{(n-1)m+1}, \cdots, \lambda_n\zeta_{nm}).$$

Then E is an f-module over $Z(M)$. Define the Riesz homomorphism $\chi : L \to E$ by

$$\chi(x) = (\xi_1, \cdots, \xi_m, \xi_1, \cdots, \xi_m, \cdots, \xi_1, \cdots, \xi_m)$$

for all $x = (\xi_1, \cdots, \xi_m)$. We claim that (E, χ) is the minimal (and maximal) Maharam extension space for $\mathcal{L}_b(L, M)$. Indeed, if $T \in \mathcal{L}_b(L, M)$ has matrix

(τ_{ij}), then the linear operator $\widehat{T}: E \to M$ with matrix

$$\begin{pmatrix} \tau_{11} & \cdots & \tau_{1m} & 0 & \cdots & 0 & \cdots\cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \tau_{21} & \cdots & \tau_{2m} & \cdots\cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots\cdots & \tau_{n1} & \cdots & \tau_{nm} \end{pmatrix}$$

with respect to the standard bases, is the unique $Z(M)$-linear operator with $T = \widehat{T} \circ \chi$. We leave the simple details to the reader. We note that in this case the space E can also be identified with the tensor product $\mathbb{R}^n \otimes \mathbb{R}^m$ (cf. Remark 5.1).

Example B

In this example we discuss Maharam extensions of kernel operators. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. We denote by $L_0(\mu) = L_0(X, \mathcal{A}, \mu)$ the space of all realvalued $\mathcal{A}$-measurable functions on X, with the usual identification of μ-a.e. equal functions. Furthermore, $\mathcal{A} \otimes \mathcal{B}$ is the product σ-algebra of $\mathcal{A}$ and $\mathcal{B}$ in $X \times Y$ and $\mu \otimes \nu$ denotes the product measure. For $g \in L_0(\mu)$ and $f \in L_0(\nu)$ we define the function $g \otimes f$ on $X \times Y$ by $(g \otimes f)(x, y) = g(x)f(y)$ $\mu \otimes \nu$-a.e.

Suppose that $L \subseteq L_0(\nu)$ and $M \subseteq L_0(\mu)$ are order dense ideals (so the carriers of L and M are Y and X respetively). Recall that a linear operator $T: L \to M$ is called an absolute kernel operator if there exists a function $T(x, y)$ in $L_0(\mu \otimes \nu)$ such that

$$\int_Y |T(\cdot, y)f(y)| \, d\nu(y) \in M \quad \forall f \in L; \tag{6.1}$$

$$Tf(x) = \int_Y T(x, y)f(y) d\nu(y) \quad \mu\text{-a.e. on } X, \quad \forall f \in L \tag{6.2}$$

(see [13], Chapter 13). As well known, $T \geq 0$ if and only if $T(x, y) \geq 0$ $\mu \otimes \nu$-a.e. on $X \times Y$. The collection of all absolute kernel operators from L into M is denoted by $\mathcal{L}_k(L, M)$. It is clear that $\mathcal{L}_k(L, M) \subseteq \mathcal{L}_n(L, M)$ and it is known that $\mathcal{L}_k(L, M)$ is an ideal, and actually a band in $\mathcal{L}_n(L, M)$ (see [13], Theorems 94.2 and 94.5). However, it may be of some interest to point out that we will not use the ideal property, but this will be an immediate consequence of the construction presented below.

Without loss of generality we may assume that ${}^{\perp}(L_n^{\sim}) = \{0\}$(cf. [5]). Now define the ideal E in $L_0(\mu \otimes \nu)$ by

$$E = \{ h \in L_0(\mu \otimes \nu) : |h| \leq \mathbb{1}_X \otimes f \text{ for some } 0 \leq f \in L \}.$$

As usual we identify $Z(M)$ with $L_\infty(\mu)$, so $p \in L_\infty(\mu)$ corresponds with $\pi_p \in Z(M)$ given by $\pi_p(g) = pg$ for all $g \in M$. For $p \in L_\infty(\mu)$ and $h \in E$ we define

$p \cdot h = (p \otimes \mathbb{I}_Y)h$, which gives E the structure of a Dedekind complete f-module over $Z(M)$. The Riesz homomorphism $\chi: L \to E$ is defined by $\chi(f) = \mathbb{I}_X \otimes f$ for all $f \in L$. For $T \in \mathcal{L}_k(L, M)$ given by (6.2) we define

$$(\widehat{T}h)(x) = \int_Y T(x, y)h(x, y)\mathrm{d}\nu(y) \quad \mu\text{-a.e. on } X$$

for all $h \in E$. It follows from (6.1) and the definition of E that $\widehat{T}h$ is well-defined and that $\widehat{T}h \in M$. Hence $\widehat{T} \in \mathcal{L}_n(E, M)$ and $\widehat{T} \circ \chi = T$. Moreover, if $p \in L_\infty(\mu)$ and $h \in E$, then

$$\begin{aligned} \widehat{T}(p \cdot h)(x) &= \int_Y T(x, y)p(x)h(x, y)\mathrm{d}\nu(y) = \\ &= p(x)\int_Y T(x, y)h(x, y)\mathrm{d}\nu(y) = p(x)\widehat{T}h(x) \end{aligned}$$

μ-a.e. on X, which shows that $\widehat{T} \in \mathcal{L}_n^{Z(M)}(E, M)$.

LEMMA 6.1 $\{\widehat{T}: 0 \le T \in \mathcal{L}_k(L, M)\}$ *separates the points of* E.

Proof. Let $0 < h \in E$ be given and put $G = \{(x, y) \in X \times Y: h(x, y) > 0\}$. Since the carrier of M is X, there exist mutually disjoint $A_n \in \mathcal{A}$ $(n = 1, 2, \cdots)$ such that $X = \bigcup_{n=1}^\infty A_n$ and $\mathbb{I}_{A_n} \in M$. Similarly, since ${}^\perp(L_n^\sim) = \{0\}$, there exist mutually disjoint $B_n \in \mathcal{B}$ $(n = 1, 2, \cdots)$ such that $Y = \bigcup_{n=1}^\infty B_n$ and $\mathbb{I}_{B_n} \in L_n^\sim$ (identifying $L_n^\sim$ with an ideal in $L_0(\nu)$). Now $\mu \otimes \nu(G) > 0$ implies that there exists n such that $\mu \otimes \nu(G \cap (A_n \times B_n)) > 0$. Let $0 \le T \in \mathcal{L}_k(L, M)$ be the operator with kernel $T(x, y) = \mathbb{I}_{A_n}(x)\mathbb{I}_{B_n}(y)$. Then $\widehat{T}h > 0$. □

LEMMA 6.2 *If* $\Lambda \in \mathcal{L}_n^{Z(M)}(E, M)$ *such that* $\Lambda \circ \chi = 0$*, then* $\Lambda = 0$.

Proof. First observe that if $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with $\mathbb{I}_B \in L$, then

$$\Lambda(\mathbb{I}_{A\times B}) = \Lambda(\mathbb{I}_A \otimes \mathbb{I}_B) = \mathbb{I}_A \cdot \Lambda(\chi\,\mathbb{I}_B) = 0.$$

Now fix $B_0 \in \mathcal{B}$ with $\mathbb{I}_{B_0} \in L$ and consider the collection

$$\mathfrak{S}_0 = \left\{ G \in \mathcal{A} \otimes \mathcal{B}: G \subseteq X \times B_0, \quad \Lambda(\mathbb{I}_G) = 0 \right\}.$$

From the above observation it follows that all sets of the form $\bigcup_{i=1}^n A_i \times B_i$, with $A_i \in \mathcal{A}$ and $B_i \in \mathcal{B}$ such that $B_i \subseteq B_0$, belong to $\mathfrak{S}_0$. Moreover, the order continuity of Λ implies that $\mathfrak{S}_0$ is a monotone class of subsets of $X \times B_0$. Consequently $\mathfrak{S}_0 = \{G \in \mathcal{A} \otimes \mathcal{B}: G \subseteq X \times B_0\}$. Since the carrier of L is equal to Y, there exist $B_n \in \mathcal{B}$ $(n = 1, 2, \cdots)$ such that $\mathbb{I}_{B_n} \in L$ and $B_n \uparrow B$. Take $G \in \mathcal{A} \otimes \mathcal{B}$ such that $\mathbb{I}_G \in E$. Then $\mathbb{I}_{G\cap B_n} \uparrow \mathbb{I}_G$ and by the above $\Lambda(\mathbb{I}_{G\cap B_n}) = 0$ for all n. Since Λ is order continuous, it follows that $\Lambda(\mathbb{I}_G) = 0$. Using once more that Λ is order continuous, we may conclude that $\Lambda = 0$. □

The above lemmas in combination with Proposition 5.13 show that (E, χ) is the minimal Maharam extension space of the ideal $\mathcal{J}$ generated by $\mathcal{L}_k(L, M)^+$. We note that this immediately implies that $\mathcal{L}_k(L, M)$ is an ideal in $\mathcal{L}_n(L, M)$. Indeed, suppose that $S \in \mathcal{J}$. Then $|S| \leq T$ for some $0 \leq T \in \mathcal{L}_k(L, M)$. Consequently, $|\widehat{S}| \leq \widehat{T}$. Now it follows from Proposition 3.10 that there exists $\pi \in Z(E)$ such that $\widehat{S} = \widehat{T} \circ \pi$ (and $|\pi| \leq I$). Identifying $Z(E)$ with $L_\infty(\mu \otimes \nu)$ we have $\pi h = mh$ for all $h \in E$ and some $m \in L_\infty(\mu \otimes \nu)$. Denoting the kernel of T by $T(x, y)$ we find that

$$\widehat{S}h(x) = \widehat{T}(mh)(x) = \int_Y T(x, y)m(x, y)h(x, y)\mathrm{d}\nu(y)$$

μ-a.e. for all $h \in E$. Hence

$$Sf(x) = \widehat{S}(\mathbb{I}_X \otimes f)(x) = \int_Y T(x, y)m(x, y)f(y)\mathrm{d}\nu(y)$$

μ-a.e. on X for all $f \in L$, which shows that $S \in \mathcal{L}_k(L, M)$. We may conclude that $\mathcal{L}_k(L, M) = \mathcal{J}$, i.e., $\mathcal{L}_k(L, M)$ is an ideal in $\mathcal{L}_n(L, M)$. That $\mathcal{L}_k(L, M)$ is actually a band now follows as in the proof of Theorem 94.5 in [13]. Furthermore we note that if $T \in \mathcal{L}_k(L, M)$ then it follows from Proposition 3.10 that there exist $G_1, G_2 \in \mathcal{A} \otimes \mathcal{B}$ such that $G_1 \cap G_2 = \emptyset$ and

$$\begin{aligned}
\widehat{T}^+h(x) &= \int_Y T(x, y)\mathbb{I}_{G_1}(x, y)h(x, y)\mathrm{d}\nu(y), \\
\widehat{T}^-h(x) &= \int_Y T(x, y)\mathbb{I}_{G_2}(x, y)h(x, y)\mathrm{d}\nu(y)
\end{aligned}$$

μ-a.e. on X for all $h \in E$. Since $\widehat{T}^+ = (T^+)^\wedge$ and $\widehat{T}^- = (T^-)^\wedge$, this implies that the kernels of T^+ and T^- are $T(x, y)^+$ and $T(x, y)^-$ respectively (cf. [13], Theorem 94.3).

Since $\mathcal{L}_k(L, M)$ is a band in $\mathcal{L}_n(L, M)$, it follows from Theorem 5.4 that the mapping $T \mapsto \widehat{T}$ is a Riesz isomorphism from $\mathcal{L}_k(L, M)$ onto $\mathcal{L}_n^{Z(M)}(E, M)$. In particular, if $\Lambda \in \mathcal{L}_n^{Z(M)}(E, M)$ then there exists a $T(x, y)$ in $L_0(\mu \otimes \nu)$ satisfying (6.1) such that

$$\Lambda h(x) = \int_Y T(x, y)h(x, y)\mathrm{d}\nu(y) \quad \mu\text{-a.e. on } X$$

for all $h \in E$.

If $0 \leq T \in \mathcal{L}_k(L, M)$, then $\widehat{T}$ is the unique operator in $\mathcal{L}_n^{Z(M)}(E, M)$ satisfying $T = \widehat{T} \circ \chi$. In particular, $\widehat{T}$ is a Maharam operator. However, in general there are more Maharam operators $0 \leq S \in \mathcal{L}_n(E, M)$ such that $T = S \circ \chi$. Indeed, let $\tau : X \to X$ be any automorphism of $(X, \mathcal{A}, \mu)$, i.e., both τ and τ^{-1} are measure preserving. Now define $0 \leq S \in \mathcal{L}_n(E, M)$ by

$$Sh(x) = \int_Y T(x, y)h(\tau x, y)\mathrm{d}\nu(y) \quad \mu\text{-a.e. on } X$$

for all $h \in E$. It is clear that $T = S \circ \chi$ and it is easy to see that S is a Maharam operator. But S is not $Z(M)$-linear with respect to the f-module structure on E which we consider above.

6.1. **Example C**

This example is concerned with Maharam extensions of Riesz homomorphisms. We start with the following general observation.

PROPOSITION 6.3 *Let L and M be Archimedean Riesz spaces with M Dedekind complete and suppose that (E, χ) is a Maharam extension space for the ideal $\mathcal{J} \subseteq \mathcal{L}_b(L, M)$. If $0 \leq T \in \mathcal{J}$ is a Riesz homomorphism, then the corresponding operator $\widehat{T} \in \mathcal{L}_n^{Z(M)}(E, M)$ is a Riesz homomorphism as well.*

Proof. It is sufficient to show that $0 \leq \Lambda \leq \widehat{T}$ in $\mathcal{L}_b(E, M)$ implies that $\Lambda = \pi\widehat{T}$ for some $0 \leq \pi \in Z(M)$ (see e.g. [3], Theorem 8.16). The mapping $S \mapsto \widehat{S}$ is a Riesz isomorphism from $\mathcal{J}$ onto an ideal in $\mathcal{L}_n^{Z(M)}(E, M)$. So there exists $0 \leq S \leq T$ such that $\Lambda = \widehat{S}$. Since T is a Riesz homomorphism it follows from Kutateladze's theorem that $S = \pi T$ for some $0 \leq \pi \in Z(M)$. Now $\pi\widehat{T} \in \mathcal{L}_n^{Z(M)}(E, M)$ and $(\pi\widehat{T}) \circ \chi = S$, so by the uniqueness it follows that $\Lambda = \widehat{S} = \pi\widehat{T}$. □

We first consider the simple case of the Maharam extension space of a single Riesz homomorphism $T_0 \colon L \to M$. As before M is assumed to be Dedekind complete. Let E be the ideal generated by $T_0(L)$ in M and $\chi = T_0$. We consider E as an f-module over $Z(M)$ with the module structure inherited from M, i.e., $\pi \cdot h = \pi(h)$ for all $h \in E$ and $\pi \in Z(M)$. Define $\widehat{T_0}$ to be the inclusion mapping of E into M. It is clear that $0 \leq \widehat{T_0} \in \mathcal{L}_n^{Z(M)}(E, M)$ and $T_0 = \widehat{T_0} \circ \chi$. Suppose that $\Lambda \in \mathcal{L}_n^{Z(M)}(E, M)$ is such that $T_0 = \Lambda \circ \chi$. If $g \in E$, then $|g| \leq T_0 u$ for some $0 \leq u \in L$, so $|\Lambda g| \leq \Lambda T_0 u = T_0 u$. Hence $\Lambda g \in E$. Now consider $\Lambda \colon E \to E$ and suppose that $h_1 \wedge h_2 = 0$ in E. Let P be the band projection in M onto $\{h_1\}^{dd}$. Then $\Lambda h_1 = \Lambda(Ph_1) = P\Lambda h_1$, and so $(\Lambda h_1) \wedge h_2 = 0$. Hence $\Lambda \in \mathrm{Orth}(E)$. Since $\Lambda g = g$ for all $g \in T_0(L)$, it now follows that $\Lambda h = h$ for all $h \in E$ and so $\Lambda \colon E \to M$ is the inclusion mapping, i.e., $\Lambda = \widehat{T_0}$. Obviously $\widehat{T_0} h > 0$ for all $0 < h \in E$, hence it follows from Corollary 5.14 that (E, χ) is the minimal Maharam extension space for the ideal $\mathcal{J}_{T_0}$ generated by T_0 in $\mathcal{L}_b(L, M)$, and for $\{T_0\}^{dd}$ as well. If in the above we take, instead of the ideal E, the band F generated by $T_0(L)$ in M then we get the maximal Maharam extension space for $\mathcal{J}_{T_0}$ (but F is in general not an extension space for $\{T_0\}^{dd}$).

Next we will discuss Maharam extension spaces for the ideal (or band) generated by a collection of Riesz homomorphisms. For the sake of simplicity we will consider the following situation. Suppose that $L \subseteq L_0(\nu)$ and $M \subseteq L_0(\mu)$ are as in Example B, and let $\{T_n \colon n = 1, 2, \cdots\}$ be an at most countable collection of Riesz homomorphisms from L into M. We assume that every T_n is a weighted composition operator, i.e., there exists $0 \le w_n \in L_0(\mu)$ and an $(\mathcal{A}, \mathcal{B})$-measurable null preserving mapping $\tau_n \colon X_n \to Y$, where $X_n = \{x \in X \colon w_n(x) > 0\}$, such that

$$T_n f(x) = w_n(x) f(\tau_n x) \quad \mu\text{-a.e. on } X_n$$

and $T_n f(x) = 0$ μ-a.e. on $X \backslash X_n$ for all $f \in L$.

For $n = 1, 2, \cdots$ we define the $(\mathcal{A}, \mathcal{A} \otimes \mathcal{B})$-measurable mapping $\widehat{\tau}_n \colon X_n \to X \times Y$ by $\widehat{\tau}_n(x) = (x, \tau_n x)$. The measure λ_n on $\mathcal{A} \otimes \mathcal{B}$ is defined by $\lambda_n(G) = \mu(\widehat{\tau}_n^{-1} G)$ for all $G \in \mathcal{A} \otimes \mathcal{B}$. For every $A \in \mathcal{A}$ we have

$$\lambda_n(A \times Y) = \mu(A \cap X_n) \le \mu(A),$$

which shows in particular that λ_n is σ-finite. Define the σ-finite measure λ on $\mathcal{A} \otimes \mathcal{B}$ by $\lambda = \sum_{n=1}^{\infty} 2^{-n} \lambda_n$. It is easy to see that the natural projection $p_1 \colon X \times Y \to X$, given by $p_1(x, y) = x$, is null preserving (with respect to λ and μ). Consequently, the mapping $g \mapsto g \otimes \mathbb{I}_Y = g \circ p_1$ is a well-defined f-algebra homomorphism from $L_0(\mu)$ into $L_0(\lambda)$. Similarly $p_2 \colon X \times Y \to Y$, defined by $p_2(x, y) = y$, is null preserving and therefore the mapping $f \mapsto \mathbb{I}_X \otimes f = f \circ p_2$ is a well-defined f-algebra homomorphism from $L_0(\nu)$ into $L_0(\lambda)$ as well. Define the ideal E in $L_0(\lambda)$ by

$$E = \left\{ h \in L_0(\lambda) \colon |h| \le \mathbb{I}_X \otimes f \text{ for some } 0 \le f \in L \right\}$$

and let $\chi \colon L \to E$ be the Riesz homomorphism given by $\chi f = \mathbb{I}_X \otimes f$ for all $f \in L$. For $m \in L_\infty(\mu)$ and $h \in E$ define $m \cdot h = (m \otimes \mathbb{I}_Y) h$. Then E is a Dedekind complete f-module over $L_\infty(\mu) = Z(M)$. We will show that (E, χ) is the minimal Maharam extension space for the ideal $\mathcal{J}$ generated by $\{T_n\}_{n=1}^{\infty}$ in $\mathcal{L}_n(L, M)$. To this end observe that it follows from the definitions that $\widehat{\tau}_n \colon X_n \to X \times Y$ is null preserving (with respect to μ and λ). Hence, for $h \in L_0(\lambda)$ we can define $\widehat{T}_n h \in L_0(\mu)$ by

$$\widehat{T}_n h(x) = w_n(x) h(\widehat{\tau}_n x) \quad \mu\text{-a.e. on } X_n$$

and $\widehat{T}_n h(x) = 0$ for $x \in X \backslash X_n$. Moreover, $h \in E$ implies that $\widehat{T}_n h \in M$, so $0 \le \widehat{T}_n \in \mathcal{L}_n(E, M)$ and it is clear that $T_n = \widehat{T}_n \circ \chi$. If $m \in L_\infty(\mu)$ and $h \in E$, then $(m \cdot h)(\widehat{\tau}_n x) = m(x) h(\widehat{\tau}_n x)$ μ-a.e. on X_n, from which it is clear that $\widehat{T}_n$ is $Z(M)$-linear. If $0 \le h \in E$ is such that $\widehat{T}_n h = 0$, then $h(\widehat{\tau}_n x) = 0$ μ-a.e. on X_n, so

$$\lambda_n \left\{ (x, y) \colon h(x, y) > 0 \right\} = \mu \left\{ x \in X_n \colon h(\widehat{\tau}_n x) > 0 \right\} = 0,$$

i.e., $h = 0$ λ_n-a.e. on $X \times Y$. Consequently, if $0 \le h \in E$ such that $\widehat{T}_n h = 0$ for all n, then $h = 0$ λ-a.e. This shows that $\{\widehat{T}_n \colon n = 1, 2, \cdots\}$ separates the points of E. The uniqueness property of $\widehat{T}_n$ is an immediate consequence of a result similar to Lemma 6.2, the proof of which in the present situation is the same. Therefore, it follows from Proposition 5.13 that (E, χ) is the minimal Maharam extension space for the ideal $\mathcal{J}$ and, by Proposition 5.12 (1), for the band $\{T_n : n = 1, 2, \cdots\}^{dd}$ as well.

7. Properties of Maharam Extension Spaces

In this section we discuss some additional properties of Maharam extension spaces. In particular we will consider the structure of the Boolean algebra of band projections in these spaces. First we recall some terminology. A net $\{f_\alpha\}$ in the Archimedean Riesz space E is called order convergent to $f \in E$, denoted by $f_\alpha \to^{(o)} f$, if there exists a net $u_\alpha \downarrow 0$ in E such that $|f_\alpha - f| \leq u_\alpha$ for all α (see, e.g., [2], Section 1). A subset D of E is called order closed if for any net $\{f_\alpha\}$ in D and $f \in E$, it follows from $f_\alpha \to^{(o)} f$ that $f \in D$. The order closed sets are the closed sets for a topology, which is called the order topology on E, denoted by τ_0. The closure of a set $D \subseteq E$ for this topology is denoted by $\overline{D}$ and is called the order closure of D. If $f_0 \in E$, then the mappings $f \mapsto f_0 + f$, $f \mapsto f_0 \vee f$ and $f \mapsto f_0 \wedge f$ are τ_0-continuous. Moreover, if K is a Riesz subspace of E, then it is not difficult to see that $\overline{K}$ is a Riesz subspace as well.

Recall that a Riesz subspace K is called regularly embedded in E if it follows from $\{f_\alpha\} \subseteq K$ and $f_\alpha \downarrow 0$ in K that $f_\alpha \downarrow 0$ in E.

LEMMA 7.1 *Suppose that K is a Riesz subspace of the Dedekind complete Riesz space E.*

(i) If K is order closed in E, then K is regularly embedded and K is Dedekind complete.

(ii) K is order closed if and only if it follows from $0 \leq f_\alpha \in K$, $f \in E$ and $f_\alpha \uparrow f$ in E that $f \in K$.

(iii) Suppose that $D \subseteq E$ such that $K \subseteq D$ and that D is closed for monotone convergence (i.e., $f_\alpha \in D$, $f \in E$ and $f_\alpha \uparrow f$, or $f_\alpha \downarrow f$, in E implies $f \in D$), then $\overline{K} \subseteq D$.

Proof. The proof of (i) is straightforward.

(ii) Assume that $0 \leq f_\alpha \in K$, $f \in E$ and $f_\alpha \uparrow f$ in E implies that $f \in E$, i.e., K is closed for monotone convergence. Now take $f_\alpha \in K$ and $f \in E$ such that $f_\alpha \to^{(o)} f$. There exist $u_\alpha \downarrow 0$ in E such that $|f - f_\alpha| \leq u_\alpha$ for all α. We may assume that $u_{\alpha_0} \geq u_\alpha$ for all α, so $\{f_\alpha\}$, is order bounded in E. For $\beta \in \{\alpha\}$ let $g_\beta = \sup_{\alpha \geq \beta} f_\alpha$. Since K is a Riesz subspace closed for monotone convergence, it is clear that $g_\beta \in K$. Now $g_\beta \downarrow f$ implies that $f \in K$, which shows that K is order closed. The converse implication is obvious.

(iii) Let D_0 be the intersection of all subsets D of E which are closed for monotone convergence with $K \subseteq D$. Clearly D_0 is closed for monotone convergence. We claim that D_0 is a Riesz subspace. For this purpose define

$$M_1 = \{ g \in E : g + f \in D_0 \quad \forall f \in K \}.$$

Then $K \subseteq M_1$ and M_1 is closed for monotone convergence. Hence $D_0 \subseteq M_1$. Now we define

$$M_2 = \{ g \in E : g + f \in D_0 \quad \forall f \in D_0 \}.$$

By the above $K \subseteq D_0$ and M_2 is closed for monotone convergence. Hence $D_0 \subseteq M_2$. This shows that $f + g \in D_0$ for all $f, g \in D_0$. In the same way it follows that $\alpha f,\ f \vee g,\ f \wedge g \in D_0$ for all $f, g \in D_0$ and $\alpha \in \mathbb{R}$, which proves the claim. Now it follows from (ii) that D_0 is order closed and consequently $K \subseteq D_0$ (and actually $\overline{K} = D_0$). □

Note that (iii) above shows that the order closure of a Riesz subspace of a Dedekind complete space is equal to the 'monotone closure'.

Now let L and M be Archimedean Riesz spaces with M Dedekind complete and suppose that (E, χ) is a Maharam extension space for the ideal $\mathcal{J} \subseteq \mathcal{L}_b(L, M)$. As before, we denote $\chi f = I \otimes f$ and $\pi \cdot \chi f = \pi \otimes f$ for all $f \in L$ and $\pi \in Z(M)$. Recall that the linear span in E of $\{\pi \otimes f : \pi \in Z(M),\ f \in L\}$ is denoted by $Z(M) \otimes_{\mathcal{J}} L$. Now we consider the smaller subspace

$$\mathcal{P}(M) \otimes_{\mathcal{J}} L = \left\{ \sum_{i=1}^{n} P_i \otimes f_i : P_i \in \mathcal{P}(M),\ f_i \in L,\ i = 1, \cdots, n;\ n \in \mathbb{N} \right\},$$

where $\mathcal{P}(M)$ denotes the Boolean algebra of all band projections in M. It is easy to see that every element op $\mathcal{P}(M) \otimes_{\mathcal{J}} L$ can we written as $\sum_{i=1}^{n} P_i \otimes f_i$ with $P_1, \cdots, P_n$ mutually disjoint in $\mathcal{P}(M)$; then the elements $P_1 \otimes f_1, \cdots, P_n \otimes f_n$ are mutually disjoint in E and

$$\left| \sum_{i=1}^{n} P_i \otimes f_i \right| = \sum_{i=1}^{n} P_i \otimes |f_i|$$

(cf. Lemma 5.2). This shows that $\mathcal{P}(M) \otimes_{\mathcal{J}} L$ is actually a Riesz subspace of E.

THEOREM 7.2 *If (E, χ) is a Maharam extension space of the ideal $\mathcal{J} \subseteq \mathcal{L}_b(L, M)$, then the order closure of $\mathcal{P}(M) \otimes_{\mathcal{J}} L$ in E is equal to E.*

Proof. The ideal generated by $\mathcal{P}(M) \otimes_{\mathcal{J}} L$ in E is the minimal Maharam extension space of $\mathcal{J}$. Moreover, this ideal is order dense in E. Hence, without loss of generality we may assume that E is minimal, i.e., $E = Z(M) \widetilde{\otimes}_{\mathcal{J}} L$.

Let F be the order closure of $\mathcal{P}(M) \otimes_{\mathcal{J}} L$ in E. By Lemma 7.1(i), F is a Dedekind complete and regularly embedded Riesz subspace of E. Observe that $\pi \otimes f \in F$ for all $\pi \in Z(M)$ and $f \in L$. Indeed, if $\pi \in Z(M)$ then for every $\epsilon > 0$ there exist mutually disjoint $P_j \in \mathcal{P}(M)$ and $\lambda_j \in \mathbb{R}\,(j = 1, \cdots, n)$ such that

$$\left| \sum_{j=1}^{n} \lambda_j P_j - \pi \right| \leq \epsilon I.$$

Hence,

$$\left|\sum_{j=1}^{n}\lambda_j P_j\otimes f-\pi\otimes f\right|=\left|\sum_{j=1}^{n}\lambda_j P_j-\pi\right|\otimes|f|\leq\epsilon(I\otimes|f|).$$

Therefore $\pi\otimes f$ belongs to the relative uniform closure of $\mathcal{P}(M)\otimes_{\mathcal{J}}L$, in particular $\pi\otimes f\in F$. Consequently, $Z(M)\otimes_{\mathcal{J}}L\subseteq F$. Next we show that F is a $Z(M)$-submodule of E. Let

$$D=\{h\in E\colon \pi\cdot h\in F \text{ for all } \pi\in Z(M)\}.$$

From the above it follows that $\mathcal{P}(M)\otimes_{\mathcal{J}}L\subseteq D$ and it is easy to see that D is order closed. Hence $F\subseteq D$, i.e., $\pi\cdot h\in F$ for all $\pi\in Z(M)$ and $h\in F$. Therefore F is an f-module over $Z(M)$.

We claim that (F,χ) is a minimal Maharam extension space of $\mathcal{J}$. For $T\in\mathcal{J}$ let $\widehat{T}$ denote the corresponding operator in $\mathcal{L}_n^{Z(M)}(E,M)$ satisfying $T=\widehat{T}\circ\chi$. Now define $\widetilde{T}=\widehat{T}_{|F}$. It is clear that $\widetilde{T}\colon F\to M$ is $Z(M)$-linear and satifies $T=\widetilde{T}\circ\chi$. Since F is regularly embedded in E it follows that $\widetilde{T}\in\mathcal{L}_n^{Z(M)}(F,M)$. Clearly $\{\widetilde{T}\colon 0\leq T\in\mathcal{J}\}$ separates the points of F, so only the uniqueness property of the operators $\widetilde{T}$ remains to be proved. To this end suppose that $\Lambda\in\mathcal{L}_n^{Z(M)}(F,M)$ is such that $\Lambda\circ\chi=0$. Then $\Lambda(P\otimes f)=P\cdot\Lambda(I\otimes f)=P\cdot\Lambda(\chi f)=0$ for all $P\in\mathcal{P}(M)$ and $f\in L$. Hence $\Lambda(h)=0$ for all $h\in\mathcal{P}(M)\otimes_{\mathcal{J}}L$. The subspace

$$N=\{h\in F\colon \Lambda(h)=0\}$$

is closed for monotone convergence in E. Indeed, suppose that $h_\alpha\in N$ and $h\in E$ such that $h_\alpha\uparrow h$ in E. Since F is order closed it follows that $h\in F$ and $h_\alpha\uparrow h$ in F, as F is regularly embedded in E. Now the order continuity of Λ implies that $\Lambda(h_\alpha)\to^{(o)}\Lambda(h)$ in M, hence $\Lambda(h)=0$, i.e., $h\in N$. For decreasing nets the corresponding result is now clear as well. Since $\mathcal{P}(M)\otimes_{\mathcal{J}}L\subseteq N$, it follows from Lemma 7.1 (iii) that $F=N$, i.e., $\Lambda=0$. Consequently, for each $T\in\mathcal{J}$ the operator $\widetilde{T}$ is unique in $\mathcal{L}_n^{Z(M)}(F,M)$ with $T=\widetilde{T}\circ\chi$. This shows that (F,χ) is a minimal Maharam extension space of $\mathcal{J}$.

Now it follows from Theorem 5.9 that there exists a $Z(M)$-linear Riesz isomorphism γ from F onto E such that $\gamma\circ\chi=\chi$. So $\gamma(I\otimes f)=I\otimes f$ for all $f\in L$, and by the $Z(M)$-linearity of γ it follows that $\gamma(h)=h$ for all $h\in\mathcal{P}(M)\otimes_{\mathcal{J}}L$. Consider the subspace

$$K=\{h\in F\colon \gamma(h)=h\}.$$

As above it follows that K is closed for monotone convergence in E. Since $\mathcal{P}(M)\otimes_{\mathcal{J}}L\subseteq K$, it follows from Lemma 7.1 (iii) that $F=K$, i.e., $\gamma(h)=h$ for all $h\in F$. Consequently $F=E$ and by this the theorem is proved. □

Next we will discuss the structure of the Boolean algebra of band projections in Maharam extension spaces in some detail. As above, let (E,χ) be a Maharam extension space of the ideal $\mathcal{J}$ in $\mathcal{L}_b(L, M)$. Up to now we have considered E, by definition, as an f-module over $Z(M)$. However, E has also a natural module structure over $Z(L)$. First the following observation.

REMARK 7.3 *Let F be a left or right f-module over some Archimedean f-algebra A in which the unit element e is assumed to be a strong order unit as well. Let $\mathfrak{S}$ be an ideal in $\mathcal{L}_b(F, M)$, with M Dedekind complete. Then $\mathfrak{S}$ has a natural right f-module structure over A. Indeed, denoting by π_a the element of $Z(F)$ corresponding to multiplication by $a \in A$, we define $R_a \colon \mathfrak{S} \to \mathfrak{S}$ by $R_a(T) = T \circ \pi_a$ for all $T \in \mathfrak{S}$. Then $R_a \in Z(\mathfrak{S})$ and the mapping $a \mapsto R_a$ is an f-algebra homomorphism from A into $Z(\mathfrak{S})$. Setting $T \cdot a = T \circ \pi_a$ for all $T \in \mathfrak{S}$ and $a \in A$ defines a right f-module structure over A on $\mathfrak{S}$.*

Now we consider the Maharam extension space E as an ideal in $\mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$ (see Theorem 5.9). We consider L as an f-module over $Z(L)$ (see Example 4.5 (i)). Applying the above remark twice (first with $\mathfrak{S} = \mathcal{J}$, then with $\mathfrak{S} = E \subseteq \mathcal{L}_n^{Z(M)}(\mathcal{J}, M)$), it follows that E is a right f-module over $Z(L)$ with $(h \cdot \sigma)(T) = h(T \circ \sigma)$, i.e.,

$$\widehat{T}(h \cdot \sigma) = (T \circ \sigma)^\wedge(h)$$

for all $h \in E$, $\sigma \in Z(L)$ and $T \in \mathcal{J}$. Note that the Riesz homomorphism $\chi : L \to E$ is $Z(L)$-linear, i.e., $\chi(\sigma f) = \chi f \cdot \sigma$ for all $f \in L$ and $\sigma \in Z(L)$. Indeed,

$$\chi(\sigma f)(T) = T(\sigma f) = (T \circ \sigma) f = (\chi f)(T \circ \sigma) = (\chi f \cdot \sigma)(T)$$

for all $T \in \mathcal{J}$. Writing $\chi f = I \otimes f$ as before, we have

$$(\pi \otimes f) \cdot \sigma = \pi \cdot (I \otimes f) \cdot \sigma = \pi \cdot (I \otimes \sigma f) = \pi \otimes (\sigma f)$$

for all $\pi \in Z(M)$, $\sigma \in Z(L)$ and $f \in L$. For $\pi \in Z(M)$ and $\sigma \in Z(L)$ we define $\pi \otimes \sigma \in Z(E)$ by $(\pi \otimes \sigma)h = \pi \cdot h \cdot \sigma$ for all $h \in E$. The mapping $(\pi, \sigma) \mapsto \pi \otimes \sigma$ is a Riesz bimorphism from $Z(M) \times Z(L)$ into $Z(E)$. Note that $(\pi T \sigma)^\wedge = T^\wedge \circ (\pi \otimes \sigma)$ for all $\pi \in Z(M)$, $\sigma \in Z(L)$ and $T \in \mathcal{J}$.

Now we assume that both L and M are Dedekind complete. If $P \in \mathcal{P}(M)$ and $Q \in \mathcal{P}(L)$, then $P \otimes Q \in \mathcal{P}(E)$ and the mapping $(P, Q) \mapsto P \otimes Q$ is a Boolean bimorphism from $\mathcal{P}(M) \times \mathcal{P}(L)$ into $\mathcal{P}(E)$. Observe that

$$(P_1 \otimes Q_1) \wedge (P_2 \otimes Q_2) = (P_1 \wedge P_2) \otimes (Q_1 \wedge Q_2)$$

for all $P_1, P_2 \in \mathcal{P}(M)$ and $Q_1, Q_2 \in \mathcal{P}(L)$. The band projections in E which are of the form $P \otimes Q$ will be called the elementary band projections. The Boolean subalgebra of $\mathcal{P}(E)$ generated by these elementary projections will be denoted by $\mathcal{P}(M) \otimes_{\mathcal{J}} \mathcal{P}(L)$. It is easy to see that $\mathcal{P}(M) \otimes_{\mathcal{J}} \mathcal{P}(L)$ is equal to

$$\left\{ \bigvee_{i=1}^{n} P_i \otimes Q_i \colon P_i \in \mathcal{P}(M),\ Q_i \in \mathcal{P}(L);\ i = 1, \cdots, n;\ n \in \mathbb{N} \right\}.$$

Moreover, every element of $\mathcal{P}(M) \otimes_{\mathcal{J}} \mathcal{P}(L)$ *can be written as* $\bigvee_{i=1}^{n} P_i \otimes Q_i$ *with* $P_1, \cdots, P_n$ *mutually disjoint in* $\mathcal{P}(M)$ *(or* $Q_1, \cdots, Q_n$ *mutually disjoint in* $\mathcal{P}(L)$*). The projections in* $\mathcal{P}(M) \otimes_{\mathcal{J}} \mathcal{P}(L)$ *will be called the simple band projections in* E.

We recall that a subalgebra $\mathfrak{A}$ *of* $\mathcal{P}(E)$ *is called a complete subalgebra if* $P_\tau \in \mathfrak{A}$, $P \in \mathcal{P}(E)$ *and* $P_\tau \uparrow P$ *in* $\mathcal{P}(E)$ *implies that* $P \in \mathfrak{A}$. *Our aim is to show that the complete subalgebra of* $\mathcal{P}(E)$ *generated by the simple band projections is equal to* $\mathcal{P}(E)$. *For this purpose we need the following preparations.*

Let E be any Dedekind complete Riesz space and suppose that $\mathfrak{A}$ *is a subalgebra of* $\mathcal{P}(E)$. *For* $0 \leq u \in E$ *we define the subspace*

$$\mathfrak{A}(u) = \left\{ \sum_{i=1}^{n} \alpha_i P_i u \colon\ P_i \in \mathfrak{A},\ \alpha_i \in \mathbb{R};\ i = 1, \cdots, n;\ n \in \mathbb{N} \right\}.$$

Every element in $\mathfrak{A}(u)$ *can be written as* $\sum_{i=1}^{n} \alpha_i P_i u$ *with* $P_1, \cdots, P_n$ *mutually disjoint in* $\mathfrak{A}$, *and then*

$$\left| \sum_{i=1}^{n} \alpha_i P_i u \right| = \sum_{i=1}^{n} |\alpha_i|\, P_i u.$$

Hence $\mathfrak{A}(u)$ *is a Riesz subspace of E in which u is a strong order unit. We define* $\mathfrak{A}[u]$ *to be the u-uniform closure of* $\mathfrak{A}(u)$ *in* E. *It is clear that* $\mathfrak{A}[u]$ *is a Riesz subspace of E, with u as a strong order unit, and that* $\mathfrak{A}[u]$ *is uniformly complete. Every* $P \in \mathfrak{A}$ *leaves* $\mathfrak{A}(u)$ *invariant and the restriction of* P *to* $\mathfrak{A}(u)$*, denoted by* $r_0(P)$*, is a band projection in* $\mathfrak{A}(u)$. *A similar statement holds for* $\mathfrak{A}[u]$*, in which case we denote the restriction of* P *by* $r(P)$. *Obviously,* r_0 *and* r *are Boolean homomorphisms from* $\mathfrak{A}$ *into* $\mathcal{P}(\mathfrak{A}(u))$ *and* $\mathcal{P}(\mathfrak{A}[u])$ *respectively.*

LEMMA 7.4

(i) For every $0 < s \in \mathfrak{A}(u)$ *there exists* $P \in \mathfrak{A}$ *such that* $r_0(P)$ *is the band projection onto the band generated by s in* $\mathfrak{A}(u)$. *In particular,* $\mathfrak{A}(u)$ *has the principal projection property.*

(ii) r_0 *is a Boolean homomorphism from* $\mathfrak{A}$ *onto* $\mathcal{P}(\mathfrak{A}(u))$.

Proof. (i) Take $0 < s \in \mathfrak{A}(u)$ and write $s = \sum_{i=1}^{n} \alpha_i P_i u$ with $P_1, \cdots, P_n$ mutually disjoint in $\mathfrak{A}$ and $\alpha_i P_i u > 0$ for all $i = 1, \cdots, n$. Define $P = \bigvee_{i=1}^{n} P_i$. It is now easy to see that $r_0(P)$ is the band projection onto the band generated by s in $\mathfrak{A}(u)$.

(ii) Let $Q \in \mathcal{P}(\mathfrak{A}(u))$ be given. Since u is a strong order unit in $\mathfrak{A}(u)$, Q is the band projection onto the band generated by Qu. Now it follows from (i) that $Q = r_0(P)$ for some $P \in \mathfrak{A}$, which shows that r_0 is surjective. □

Now we assume in addition that $\mathfrak{A}$ is a complete subalgebra of $\mathcal{P}(E)$. The carrier projection with respect to $\mathfrak{A}$ of an element $0 \leq u \in E$ is defined by

$$P_0 = \inf\{\, P \in \mathfrak{A} \colon\ Pu = u \,\}$$

(where the infimum is taken in $\mathcal{P}(E)$). Since $\mathfrak{A}$ is a complete subalgebra of $\mathcal{P}(E)$ we have $P_0 \in \mathfrak{A}$. Note furthermore that $P_0 u = u$. Let

$$\mathfrak{A}_u = \{ P \in \mathfrak{A} \colon P \leq P_0 \}.$$

Then $\mathfrak{A}_u$ is a complete Boolean algebra with unit P_0.

LEMMA 7.5 *Assume that* $\mathfrak{A}$ *is a complete subalgebra of* $\mathcal{P}(E)$ *and* $0 \leq u \in E$.
(i) The restriction mapping r_0 *is a Boolean isomorphism from* $\mathfrak{A}_u$ *onto* $\mathcal{P}(\mathfrak{A}(u))$.
(ii) $\mathfrak{A}(u)$ *has the projection property.*

Proof. (i) Since $P_0 u = u$, it follows that $r_0(P_0) = I$, so r_0 is a Boolean homomorphism. If $P \in \mathfrak{A}$, then $Pu = (P \wedge P_0)u$ and so $r_0(P) = r_0(P \wedge P_0)$. Hence $r_0 \colon \mathfrak{A}_u \to \mathcal{P}(\mathfrak{A}(u))$ is surjective. Now suppose that $P \in \mathfrak{A}_u$ with $r_0(P) = 0$, i.e., $Pu = 0$. Then $(P_0 - P)u = u$, so by the definition of P_0 it follows that $P_0 \leq P_0 - P$. Hence $P = 0$.

(ii) It follows from (i) that $\mathcal{P}(\mathfrak{A}(u))$ is a complete Boolean algebra. By Lemma 7.5 (ii), $\mathfrak{A}(u)$ has the principal projection property and consequently, by [8], Theorem 30.6, $\mathfrak{A}(u)$ has the projection property. □

In the next proposition we collect the properties of the subspaces $\mathfrak{A}[u]$ which will be used below.

PROPOSITION 7.6 *Assume that* $\mathfrak{A}$ *is a complete subalgebra of* $\mathcal{P}(E)$ *and* $0 \leq u \in E$.
(i) $\mathfrak{A}[u]$ *is Dedekind complete and is regularly embedded in* E.
(ii) The order intervals in $\mathfrak{A}[u]$ *are order closed in* E.
(iii) The restriction mapping r *is a Boolean isomorphism from* $\mathfrak{A}_u$ *onto* $\mathcal{P}(\mathfrak{A}[u])$.

Proof. First note that $\mathfrak{A}(u)$ is strongly order dense in $\mathfrak{A}[u]$, i.e., for every $0 < f \in \mathfrak{A}[u]$ there exists $s \in \mathfrak{A}(u)$ such that $0 < s \leq f$. Indeed, take $0 < \lambda \in \mathbb{R}$ such that $(f - 2\lambda u)^+ > 0$. Then there exist $t \in \mathfrak{A}(u)$ such that $|f - t| \leq \lambda u$. Now it is easily seen that $s = (t - \lambda u)^+$ satisfies $0 < s \leq f$. Now let B be a band in $\mathfrak{A}[u]$. Since $\mathfrak{A}(u)$ is strongly order dense in $\mathfrak{A}[u]$, it follows that $B \cap \mathfrak{A}(u)$ is a band in $\mathfrak{A}(u)$ (see e.g. [13], Section 79). By Lemma 7.5, $B \cap \mathfrak{A}(u)$ is a projection band in $\mathfrak{A}(u)$and there exists $P \in \mathfrak{A}_u$ such that $r_0(P)$ is the band projection onto $B \cap \mathfrak{A}(u)$. Using again that $\mathfrak{A}(u)$ is strongly order dense in $\mathfrak{A}[u]$ we see that $P(\mathfrak{A}[u]) = B$, hence the restriction $r(P)$ is the band projection in $\mathfrak{A}[u]$ onto B. This shows that $\mathfrak{A}[u]$ has the projection property. Consequently, $\mathfrak{A}[u]$ is Dedekind complete, as $\mathfrak{A}[u]$ is uniformly complete.

Note that the above shows that the restriction mapping $r \colon \mathfrak{A}_u \to \mathcal{P}(\mathfrak{A}[u])$ is a surjective Boolean homomorphism. Since $P \in \mathfrak{A}_u$ and $Pu = 0$ implies that $P = 0$, it follows that r is an isomorphism, by which (iii) is proved.

Now we show that $\mathfrak{A}[u]$ is regularly embedded in E. Suppose that $f_\tau \downarrow 0$ in $\mathfrak{A}[u]$. Without loss of generality we may assume that $0 \leq f_\tau \leq u$. Take $\varepsilon > 0$ and let Q_τ be the band projection in $\mathfrak{A}[u]$ onto the band B_τ generated by $(f_\tau - \varepsilon u)^+$

in $\mathfrak{A}[u]$. Then $\bigcap_\tau B_\tau = \{0\}$ implies that $Q_\tau \downarrow 0$ in $\mathcal{P}(\mathfrak{A}[u])$. For each τ there exists a unique $P_\tau \in \mathfrak{A}_u$ such that $Q_\tau = r(P_\tau)$. Since r is a Boolean isomorphism it follows that $P_\tau \downarrow 0$ in $\mathfrak{A}_u$ and hence $P_\tau \downarrow 0$ in $\mathcal{P}(E)$, as $\mathfrak{A}$ is a complete subalgebra. Therefore $P_\tau u \downarrow 0$ in E. Using that

$$0 \le f_\tau = (f_\tau - \varepsilon u)^+ + f_\tau \wedge \varepsilon u \le (f_\tau - \varepsilon u)^+ + \varepsilon u$$

and that $(f_\tau - \varepsilon u)^+ \le P_\tau u$, it follows that $\inf_\tau f_\tau \le \varepsilon u$. This holds for all $\varepsilon > 0$, which shows that $f_\tau \downarrow 0$ in E.

(ii) This follows from the following general observation. If K is a Dedekind complete Riesz subspace of the Dedekind complete space E and if K is regularly embedded in E, then the order intervals in K are order closed in E. Indeed, take $0 \le w \in K$ and suppose that $f_\alpha \in K$ such that $0 \le f_\alpha \le w$ and $f_\alpha \to^{(o)} g$ in E. It is clear that $0 \le g \le w$. For $\beta \in \{\alpha\}$ define $g_\beta = \sup\{f_\alpha \colon \alpha \ge \beta\}$ in E. Since K is Dedekind complete and regularly embedded in E, it follows that $g_\beta \in K$. Now $g_\beta \downarrow g$ in E and, again using the assumptions on K, we may conclude that $g \in K$. □

In the following proposition we denote by K_u the principal ideal in a Riesz space K generated by $0 \le u \in K$.

PROPOSITION 7.7 *Let K be a Riesz subspace of the Dedekind complete space E and let $\mathfrak{A}_0$ be a subalgebra of $\mathcal{P}(E)$. Assume that*
(i) the order closure of K is equal to E;
(ii) there exists a subset $D \subseteq K^+$ which is majorizing in K^+ such that $K_u \subseteq \mathfrak{A}_0[u]$ for all $u \in D$.
Then the complete subalgebra of $\mathcal{P}(E)$ generated by $\mathfrak{A}_0$ is equal to $\mathcal{P}(E)$.

Proof. Let $\mathfrak{A}$ denote the complete subalgebra of $\mathcal{P}(E)$ generated by $\mathfrak{A}_0$. We first show that $E_u = \mathfrak{A}[u]$ for all $u \in D$. Since u is a strong order unit in $\mathfrak{A}[u]$ it is clear that $\mathfrak{A}[u] \subseteq E_u$. Fix $n \in \mathbb{N}$ and consider the set

$$A = \{ f \in E \colon |f| \wedge nu \in \mathfrak{A}[u] \}.$$

If $f \in K$, the $|f| \wedge nu \in K_u \subseteq \mathfrak{A}_0[u]$ and obviously $\mathfrak{A}_0[u] \subseteq \mathfrak{A}[u]$. Hence $K \subseteq A$. By Proposition 7.6 (ii) order intervals in $\mathfrak{A}[u]$ are order closed in E, which implies that A is order closed in E. Consequently, $E = \overline{K} \subseteq A$, i.e., $E = A$. This suffices to show that $E_u \subseteq \mathfrak{A}[u]$, hence $E_u = \mathfrak{A}[u]$.

Let $Q \in \mathcal{P}(E)$ be given. If $u \in D$, then $Qu \in E_u = \mathfrak{A}[u]$, so Qu is a component of u in $\mathfrak{A}[u]$. It follows from Proposition 7.6 (iii) that there exists a unique $P_u \in \mathfrak{A}_u$ such that $P_u(u) = Qu$. Observe that $u \le v$ in D implies that $P_u \le P_v$. Indeed, it follows from $P_v(v) = Qv$ that $P_v = Q$ on E_v and so $P_v(u) = Qu = P_u(u)$. Using the definition of $\mathfrak{A}_u$ it is now easy to see that $P_v \le P_u$. We claim that $P_u \le Q$ in $\mathcal{P}(E)$ for all $u \in D$. Indeed, take $w \in D$. Since D is majorizing in K^+ there exists $v \in D$ such that $u + w \le v$. Then $P_v = Q$ on E_v

and it follows from the above that

$$P_u(w) \leq P_v(w) = Qw.$$

Hence $P_u(w) \leq Qw$ for all $w \in D$. Since D is majorizing in K^+ and $\overline{K} = E$, this implies that $P_u \leq Q$ in $\mathcal{P}(E)$, which proves the claim.

Now define

$$P_1 = \sup\{ P_u : u \in D \}$$

in $\mathcal{P}(E)$. Since $\mathfrak{A}$ is a complete subalgebra of $\mathcal{P}(E)$ we have $P_1 \in \mathfrak{A}$, and it is clear that $P_1 \leq Q$. Moreover, $P_1 u \geq P_u(u) = Qu$ for all $u \in D$, i.e., $P_1 u = Qu$ for all $u \in D$. Consequently $P_1 = Q$ and hence $Q \in \mathfrak{A}$. This shows that $\mathfrak{A} = \mathcal{P}(E)$, by which the proposition is proved. □

Now we return to the situation where E is a Maharam extension space of an ideal $\mathcal{J}$ in $\mathcal{L}_b(L, M)$.

THEOREM 7.8 *Let L and M be Dedekind complete Riesz spaces and suppose that (E, χ) is a Maharam extension space of the ideal $\mathcal{J} \subseteq \mathcal{L}_b(L, M)$. Then $\mathcal{P}(E)$ is equal to the complete subalgebra generated by $\mathcal{P}(M) \otimes_{\mathcal{J}} \mathcal{P}(L)$.*

Proof. We will apply the above proposition with $\mathfrak{A}_0 = \mathcal{P}(M) \otimes_{\mathcal{J}} \mathcal{P}(L)$, $K = \mathcal{P}(M) \otimes_{\mathcal{J}} L$ and

$$D = \{ I \otimes f : 0 \leq f \in L \}.$$

By Theorem 7.2 we know that $\overline{K} = E$ and so it remains to show that $K_u \subseteq \mathfrak{A}_0[u]$ for all $u \in D$. To this end, fix $u = I \otimes f$ with $0 \leq f \in L$. It is clearly enough to show that $0 \leq P \otimes g \leq I \otimes f$, with $P \in \mathcal{P}(M)$ and $0 \leq g \leq f$ in L, implies that $P \otimes g \in \mathfrak{A}_0[u]$. Since $P \otimes g = (P \otimes I)(I \otimes g)$ and $P \otimes I \in \mathfrak{A}_0$, it is sufficient to prove that $I \otimes g \in \mathfrak{A}_0[u]$. Let $\varepsilon > 0$ be given. By Freudenthal's spectral theorem there exist mutually disjoint $Q_1, \cdots, Q_n \in \mathcal{P}(L)$ and $0 \leq \alpha_1, \cdots, \alpha_n \in \mathbb{R}$ such that

$$\left| \sum_{j=1}^{n} \alpha_j Q_j f - g \right| \leq \varepsilon f.$$

Then

$$\left| \sum_{j=1}^{n} \alpha_j (I \otimes Q_j f) - I \otimes g \right| \leq \varepsilon (I \otimes f) = \varepsilon u,$$

and $\sum_{j=1}^{n} \alpha_j (I \otimes Q_j f) = \sum_{j=1}^{n} \alpha_j (I \otimes Q_j) u \in \mathfrak{A}_0(u)$. This suffices to prove the theorem. □

Now we discuss some consequences of the above result. For simplicity we assume that $\mathcal{J} \subseteq \mathcal{L}_b(L, M)$ is a band, where L and M are both Dedekind complete. Let (E, χ) be a Maharam extension space of $\mathcal{J}$. For $T \in \mathcal{J}$ we denote by $\alpha(T) = \widehat{T}$ the corresponding operator in $\mathcal{L}_n^{Z(M)}(E, M)$, i.e., $\widehat{T} \circ \chi = T$. Then α is a Riesz isomorphism from $\mathcal{J}$ onto $\mathcal{L}_n^{Z(M)}(E, M)$ (see Theorems 5.9 and 5.4). For $P \in \mathcal{P}(E)$ we denote by P^* the band projection in $\mathcal{L}_n^{Z(M)}(E, M)$ defined by $P^*(\Lambda) = \Lambda \circ P$ for all $\Lambda \in \mathcal{L}_n^{Z(M)}(E, M)$. It is an immediate consequence of Proposition 3.8 that the mapping $P \mapsto P^*$ is a Boolean isomorphism from $\mathcal{P}(E)$ onto $\mathcal{P}(\mathcal{L}_n^{Z(M)}(E, M))$. Furthermore, the Riesz isomorphism α induces a Boolean isomorphism from $\mathcal{P}(\mathcal{J})$ onto $\mathcal{P}(\mathcal{L}_n^{Z(M)}(E, M))$. Combination of these observations shows that there exists a Boolean isomorphism τ_α from $\mathcal{P}(\mathcal{J})$ onto $\mathcal{P}(E)$, which is completely determined by

$$P(T)^\wedge = \widehat{T} \circ \tau_\alpha(P) \quad \forall P \in \mathcal{P}(\mathcal{J}), \ \forall T \in \mathcal{J}.$$

For $P \in \mathcal{P}(M)$ and $Q \in \mathcal{P}(L)$ we define $P \otimes Q \in \mathcal{P}(\mathcal{J})$ by $(P \otimes Q)(T) = PTQ$ for all $T \in \mathcal{J}$. We call $P \otimes Q$ an elementary band projection in $\mathcal{J}$. Note that

$$(PTQ)^\wedge(h) = \widehat{T}(P \cdot h \cdot Q) = \widehat{T}(P \otimes Q)(h)$$

for all $h \in E$, so $\tau_\alpha(P \otimes Q) = P \otimes Q$ (where $P \otimes Q \in \mathcal{P}(E)$ is as defined before). The algebra $\mathcal{U}(\mathcal{J})$ of simple band projections in $\mathcal{J}$ is defined by

$$\mathcal{U}(\mathcal{J}) = \left\{ \bigvee_{i=1}^{n} P_i \otimes Q_i : \ P_i \in \mathcal{P}(M), \ \ Q_i \in \mathcal{P}(L); \ i = 1, \cdots, n; \ n \in \mathbb{N} \right\}.$$

It is now clear that τ_α induces a Boolean algebra isomorphism from $\mathcal{U}(\mathcal{J})$ onto $\mathcal{P}(M) \otimes_{\mathcal{J}} \mathcal{P}(L)$.

These observations in combination with Theorem 7.8 immediately yield the following result.

COROLLARY 7.9 *Let L an M be Dedekind complete Riesz spaces and let $\mathcal{J}$ be a band in $\mathcal{L}_b(L, M)$. Then the complete algebra generated in $\mathcal{P}(\mathcal{J})$ by the algebra $\mathcal{U}(\mathcal{J})$ of simple band projections, is equal to $\mathcal{P}(\mathcal{J})$.*

Now we specialize the above result to the case that $\mathcal{J} = \{T\}^{dd}$ for some $0 \leq T \in \mathcal{L}_b(L, M)$. Then $\mathcal{P}(\mathcal{J})$ is Boolean isomorphic with the Boolean algebra $\mathcal{B}_T$ of components of T in $\mathcal{L}_b(L, M)$. Moreover, $\mathcal{U}(\mathcal{J})$ corresponds to the subalgebra $\mathcal{U}_T$ of $\mathcal{B}_T$ consisting of all so-called simple components of T, i.e.,

$$\mathcal{U}_T = \left\{ \bigvee_{i=1}^{n} P_i T Q_i : \ P_i \in \mathcal{P}(M), \ \ Q_i \in \mathcal{P}(L); \ i = 1, \cdots, n; \ n \in \mathbb{N} \right\}.$$

This gives the following corollary.

COROLLARY 7.10 *Let L and M be Dedekind complete Riesz spaces and $0 \leq T \in \mathcal{L}_b(L, M)$. Then the complete algebra generated by the algebra $\mathcal{U}_T$ of simple components of T is equal to the Boolean algebra $\mathcal{B}_T$ of all components.*

REMARK 7.11 *Under the assumption that ${}^\perp(M_n^\sim) = \{0\}$, the above result follows from [4], Theorem 3.10. The conclusion in the latter theorem is stronger than in Corollary 7.10. In fact, if ${}^\perp(M_n^\sim) = \{0\}$ then $\mathcal{B}_T$ can be obtained from $\mathcal{U}_T$ via an up-down process which terminates after at most three steps. However, it should be noted that in the general situation of Corollary 7.10 it follows from the theory of Boolean algebras that the complete algebra generated by $\mathcal{U}_T$ is equal to the monotone class generated by $\mathcal{U}_T$. The fact that under the hypothesis that ${}^\perp(M_n^\sim) = \{0\}$ the up-down process terminates after finitely many steps is due to the presence of sufficiently many measures on $\mathcal{B}_T$ in that case. The following corollary is in the same vein as the results in [4], Section 4.*

COROLLARY 7.12 *Let L and M be Dedekind complete Riesz spaces and suppose that G is a linear subspace of $\mathcal{L}_b(L, M)$ which satisfies:*
(i) $PTQ \in G$ for all $0 \leq T \in G$, $P \in \mathcal{P}(M)$ and $Q \in \mathcal{P}(L)$;
(ii) if $T_\alpha, T \in G$ and $S \in \mathcal{L}_b(L, M)$ such that $0 \leq T_\alpha \uparrow S \leq T$ in $\mathcal{L}_b(L, M)$, then $S \in G$.
Then G is order convex in $\mathcal{L}_b(L, M)$, i.e., $0 \leq S \leq T \in G$ implies $S \in G$.

Proof. Let $0 \leq T \in G$ be fixed and define $\mathfrak{A} = \mathcal{B}_T \cap G$. It follows from (i) that all simple components of T are contained in $\mathfrak{A}$, i.e., $\mathcal{U}_T \subseteq \mathfrak{A}$. Moreover, it follows from (ii) that $\mathfrak{A}$ is closed for monotone convergence in $\mathcal{L}_b(L, M)$. Therefore, the complete algebra generated by $\mathcal{U}_T$ is contained in $\mathfrak{A}$ and so, by Corollary 7.10, $\mathfrak{A} = \mathcal{B}_T$. Now suppose that $0 \leq S \leq T$ in $\mathcal{L}_b(L, M)$. By Freudenthal's spectral theorem there exists a sequence $\{S_n\}_{n=1}^\infty$ in $\mathcal{L}_b(L, M)$ such that $0 \leq S_n \uparrow S$ and each S_n is of the form $\sum_{j=1}^k \alpha_j R_j$ with $R_j \in \mathcal{B}_T$ and $0 \leq \alpha_j \leq 1$ $(j = 1, \cdots, n)$. The above implies that $S_n \in G$ for all n and it follows from (ii) that $S \in G$. □

The following example illustrates how certain band projections in the space of order bounded operators can be described in terms of Maharam extension spaces.

EXAMPLE 7.13 *Assume that L and M are Dedekind complete Riesz spaces. In the space of all order bounded operators from L into M we have the band decomposition*

$$\mathcal{L}_b(L, M) = \mathcal{L}_n(L, M) \oplus \mathcal{L}_s(L, M),$$

where $\mathcal{L}_n(L, M)$ denotes the band of all normal (i.e., order continuous) operators and $\mathcal{L}_s(L, M) = \mathcal{L}_n(L, M)^d$ (see e.g. [13], Section 83). The band projection in $\mathcal{L}_b(L, M)$ onto $\mathcal{L}_n(L, M)$ will be denoted by $\mathcal{P}_n$. Since L is Dedekind complete, it follows via a standard argument that an operator $0 \leq T \in \mathcal{L}_b(L, M)$ is normal if and only if $Q_\alpha \downarrow 0$ in $\mathcal{P}(L)$ implies that $TQ_\alpha \downarrow 0$ in $\mathcal{L}_b(L, M)$.

Let (E, χ) be the minimal Maharam extension space of $\mathcal{J} = \mathcal{L}_b(L, M)$ i.e., $E = Z(M)\widetilde{\otimes}_{\mathcal{J}} L$. As before, for $T \in \mathcal{L}_b(L, M)$ we denote the corresponding operator in $\mathcal{L}_n^{Z(M)}(E, M)$ by $\widehat{T}$, so $\widehat{T}(I \otimes f) = Tf$ for all $f \in L$. By the observations preceding Corollary 7.9, there exists a unique $P_n \in \mathcal{P}(E)$ such that

$$\mathcal{P}_n(T)^\wedge = \widehat{T} \circ P_n \text{ for all } T \in \mathcal{L}_b(L, M).$$

There are several descriptions of this band projection P_n. First of all, we claim that

$$P_n = \sup\{P \in \mathcal{P}(E)\colon Q_\alpha \downarrow 0 \text{ in } \mathcal{P}(L) \text{ implies } P(I \otimes Q_\alpha) \downarrow 0 \text{ in } \mathcal{P}(E)\} \tag{7.1}$$

(and the supremum is attained by P_n). Indeed, suppose that $Q_\alpha \downarrow 0$ in $\mathcal{P}(L)$. Then $\mathcal{P}_n(T)Q_\alpha \downarrow 0$ in $\mathcal{L}_b(L, M)$. Since the mapping $S \mapsto \widehat{S}$ is a Riesz isomorphism from $\mathcal{L}_b(L, M)$ onto $\mathcal{L}_n^{Z(M)}(E, M)$, this implies that

$$T^\wedge P_n(I \otimes Q_\alpha) = \mathcal{P}_n(T)^\wedge(I \otimes Q_\alpha) = [\mathcal{P}_n(T)Q_\alpha]^\wedge \downarrow 0$$

in $\mathcal{L}_n^{Z(M)}(E, M)$ for all $0 \le T \in \mathcal{L}_b(L, M)$. Since $\{\widehat{T}\colon 0 \le T \in \mathcal{L}_b(L, M)\}$ separates the points of E, it follows that $P_n(I \otimes Q_\alpha) \downarrow 0$ in $\mathcal{P}(E)$. Now assume that $P \in \mathcal{P}(E)$ is such that $P(I \otimes Q_\alpha) \downarrow 0$ in $\mathcal{P}(E)$ whenever $Q_\alpha \downarrow 0$ in $\mathcal{P}(L)$. Take $0 \le T \in \mathcal{L}_b(L, M)$ and let $0 \le S \in \mathcal{L}_b(L, M)$ be such that $\widehat{S} = \widehat{T}P$. Then $Q_\alpha \downarrow 0$ in $\mathcal{P}(L)$ implies that

$$(SQ_\alpha)^\wedge = \widehat{S}P(I \otimes Q_\alpha) \downarrow 0,$$

and hence $SQ_\alpha \downarrow 0$ in $\mathcal{L}_b(L, M)$. Consequently, $0 \le S \in \mathcal{L}_n(L, M)$, so $0 \le S \le \mathcal{P}_n(T)$. This shows that $0 \le \widehat{T}P \le \widehat{T}P_n$ for all $0 \le T \in \mathcal{L}_b(L, M)$. Hence $P \le P_n$, which proves (7.1). Note that (7.1) is equivalent to

$$P_n = \sup\{\, P \in \mathcal{P}(E)\colon u_\alpha \downarrow 0 \text{ in } L \text{ implies } P(I \otimes u_\alpha) \downarrow 0 \text{ in } E\,\}.$$

Another description of P_n is given by

$$P_n = \inf\left\{\sup_\alpha I \otimes Q_\alpha\colon Q_\alpha \uparrow I \text{ in } \mathcal{P}(L)\right\}. \tag{7.2}$$

Indeed, denote the infimum on the righthand side by P_0. If $Q_\alpha \uparrow I$ in $\mathcal{P}(L)$, then it follows from (7.1) that $P_n(I \otimes Q_\alpha) \uparrow P_n$ in $\mathcal{P}(E)$, so $P_n \le \sup_\alpha I \otimes Q_\alpha$. This shows that $P_n \le P_0$. Now suppose that $Q_\alpha \downarrow 0$ in $\mathcal{P}(L)$. Then $I - Q_\alpha \uparrow I$ in $\mathcal{P}(L)$, so it follows from the definition of P_0 that $P_0 \le \sup_\alpha I \otimes (I - Q_\alpha)$, which implies that $P_0(I \otimes Q_\alpha) \downarrow 0$ in $\mathcal{P}(E)$. Via (7.1) we may conclude that $P_0 \le P_n$ hence $P_n = P_0$.

Observe that it is an immediate consequence of (7.2) that

$$T_n = \mathcal{P}_n(T) = \inf\left\{\sup_\alpha TQ_\alpha\colon Q_\alpha \uparrow I \text{ in } \mathcal{P}(L)\right\} \tag{7.3}$$

for all $0 \le T \in \mathcal{L}_b(L, M)$, which is a well-known formula (see e.g. [3], Theorem 4.6).

8. Concluding Remarks

In the present paper we have concentrated on the construction and some properties of Maharam extension spaces in the general framework of Archimedean and Dedekind complete Riesz spaces. Therefore the results have a rather algebraic flavour, as topological or Banach lattice structure did not play any particular role. These aspects will be considered at an other occasion. We end this paper with a brief indication which topics will be discussed in a subsequent paper:

(i) Representations of $Z(M)$-modules and $Z(M)$-linear operators; this will clarify in more detail the relations between the work of D. Maharam and our results.

(ii) Maharam extension spaces for Banach lattices and Banach function spaces.

(iii) The connection between Maharam extension spaces and the theory of tensor products. Although we have used the tensor product notation frequently in the present paper, we have not yet discussed the precise ralationship between our results and the theory of tensor products of Riesz spaces (and Boolean algebras).

(iv) Maharam extensions for operators of a Riesz space into itself. This will be in particular of interest for the discussion in this framework of, for example, the powers of an operator and spectral and asymptotic properties.

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